

September 22, 2026

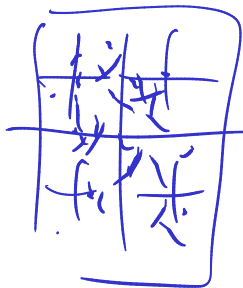
Announcements

- HV3

Goals

- Barnes - Hw
 - faster mpole fluidity
- Fast Multipole
 - adaptive $\leftarrow$
 - alt. reality
- Dual-space

Review



1
4
42

Barnes-Hut: Next Revised Cost Estimate

Summarize the algorithm (so far) and the associated cost.

		How often	Cost	Total
①	Form upoles	$N_{src} \cdot L_{leaf}$	K	$KN \log N$
②	Evaluate upoles	$N_{tgt} \cdot 27L$	K	$NK \log N$
③	Direct interactions (self + neighbors)	$C \cdot \frac{N}{m}$	m^2	$C \cdot \frac{N}{m}$

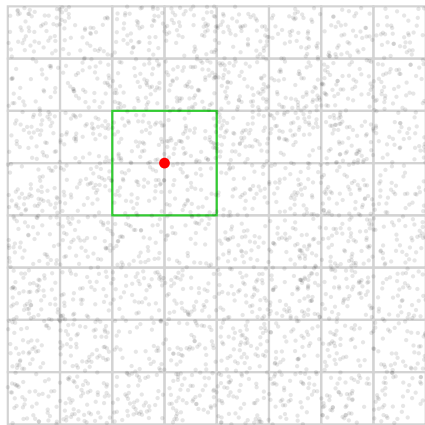
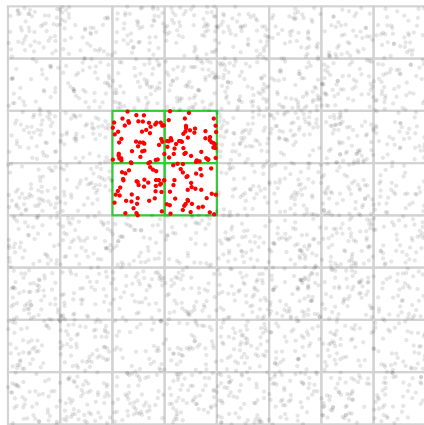
N # particles

K # terms in an expansion

m # particles / box

L # levels $\sim \log N$

Barnes-Hut: Putting Multipole Expansions to Work



(Figure following G. Martinsson)

How could this process be sped up?

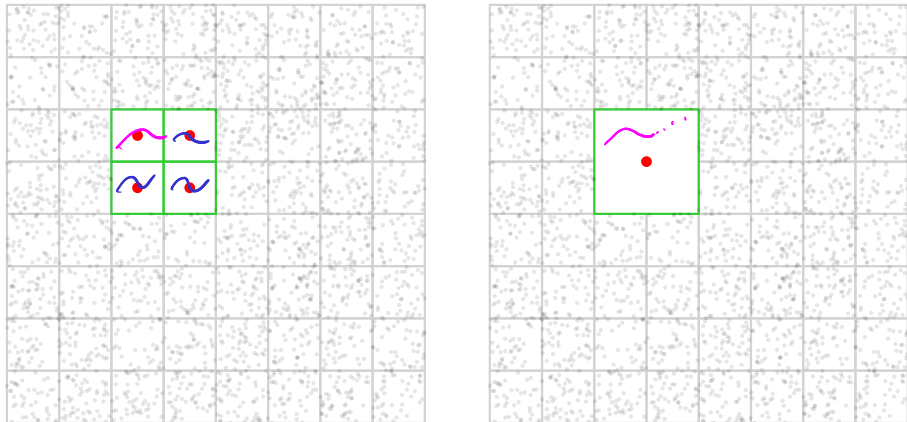
Barnes-Hut: Clumps of Boxes?

Observation: The amount of work does not really decrease as we go up the tree: Fewer boxes, but more particles in each of them.

But we already compute multipoles to summarize lower-level boxes...



Barnes-Hut: Putting Multipole Expansions to Work



(Figure following G. Martinsson)

To get a new 'big' multipole from a 'small' multipole, we need a new mathematical tool.

Barnes-Hut: Translations



A handwritten pink arrow points from the left towards the letters 'hw', which are also written in pink. The entire content is enclosed within a light gray rectangular box with a thin black border.

Cost of Multi-Level Barnes-Hut

$$\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{1 - \frac{1}{4}}$$

New construction phase:

Level	Work	Cost	How many
(lowest, leaf)	src $\rightarrow$ mpole	mK	N/m
L-1	mpole $\rightarrow$ mpole	K^2	$(N/m)/4$
L-2			$(N/m)/16$

- ① Compute mpole $NK + cK^2N$
- ② Eval mpole $N \log(N)$
- ③ close boxes $O(N)$

Better than K^2

$$K = \# \text{ terms}$$

In d dimensions : $K = O(p^d)$ if not obeying PDE

$$K = O(p^{d-1})$$

$$\sum_{|\vec{m}| \leq p} \alpha_{\vec{m}} (\vec{x} - \vec{c})^{\vec{p}} = \sum (x_1 - c_1) \underbrace{\sum_p (x_2 - c_2)}_{\substack{O(p^{d-1}) \text{ coeff} \\ O(p^d)}} \sim \text{sum factorization}$$

$\rightarrow$ Ideally, $O(p^d)$

Khanh's Barnes-local

transpose Barnes-thunk  upward



"downward"

$O(n \log n)$

Cost of Multi-Level Barnes-Hut: Observations

Observation: Multipole evaluation remains as the single most costly bit of this algorithm. *Fix?*

Idea: Exploit the tree structure also in performing this step.
If 'upward' translation of multipoles helped earlier, maybe 'downward' translation of *local* expansions can help now.

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Barnes-Hut

Fast Multipole

Ewald Summation and DMK

Direct Solvers

The Butterfly Factorization

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

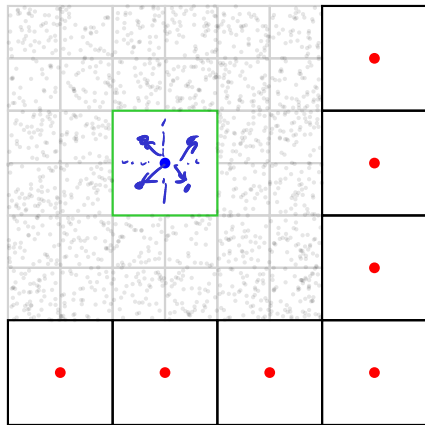
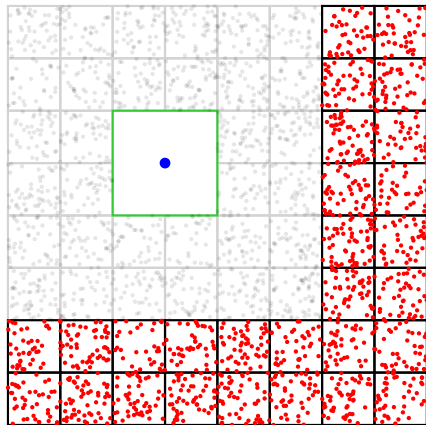
Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Using Multipole-to-Local



(Figure following G. Martinsson)

Come up with an algorithm that computes the interaction in the figure.

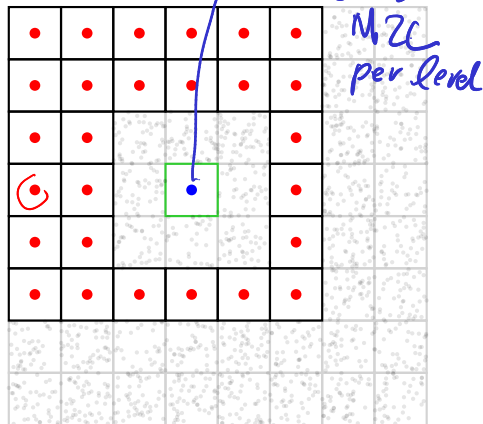
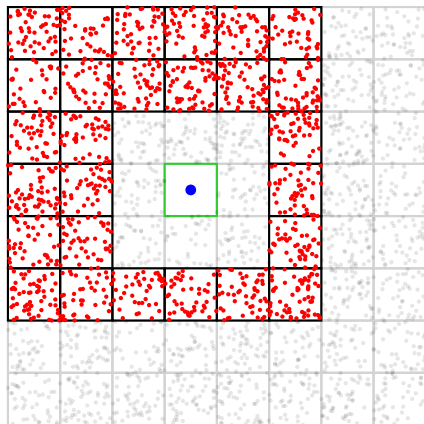
Using Multipole-to-Local

Come up with an algorithm that computes the interaction in the figure.



$$G(\vec{x}, \vec{y}) = G(\vec{x} - \vec{y}) \quad \text{"translation invariance"}$$

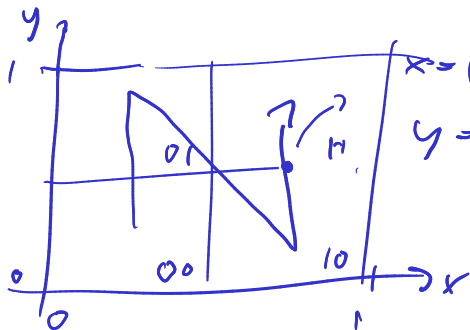
Using Multipole-to-Local: Next Level



(Figure following G. Martinsson)

Assuming we retain information from the previous level, how can we obtain a valid local expansion on the **target** box?

Morla order



$$x = (0, \overset{1}{\underset{1}{1}} - - -)_2$$

$$y = (0, \overset{1}{\underset{1}{0}} - - -)_2$$

$$(\overset{1}{\underset{1}{1}} \overset{1}{\underset{1}{0}})_2$$

Using Multipole-to-Local: Next Level

Assuming we retain information from the previous level, how can we obtain a valid local expansion on the **target** box?



Define 'Interaction List'

For a box b , the interaction list I_b consists of all boxes b' so that

- b and b' are on the same level
- b and b' don't touch
- $\text{parent}(b)$ and $\text{parent}(b')$ touch

Provide an upper bound on the number of elements of I_b .

The Fast Multipole Method ('FMM')

Upward pass

1. Build tree
2. Compute interaction lists
3. Compute lowest-level multipoles from sources
4. Loop over levels $\ell = L - 1, \dots, 2$:
 - 4.1 Compute multipoles at level ℓ by $\text{mp} \rightarrow \text{mp}$

Downward pass

1. Loop over levels $\ell = 2, 3, \dots, L - 1$:
 - 1.1 Loop over boxes b on level ℓ :
 - 1.1.1 Add contrib from l_b to local expansion by $\text{mp} \rightarrow \text{loc}$
 - 1.1.2 Add contrib from parent to local exp by $\text{loc} \rightarrow \text{loc}$
2. Evaluate local expansion and direct contrib from 9 neighbors.

Overall algorithm: Now $O(N)$ complexity.

Note: L levels, numbered $0, \dots, L - 1$. Loop indices above *inclusive*.