

October 6, 2026
Announcements

Goals

- Interpolation
- Fourier
- Ewald summation
- NUFFT
- DMK

periodic
~~high-frequency~~
adaptive

Review

- FMM with no expansions
↳ 'interpolation'

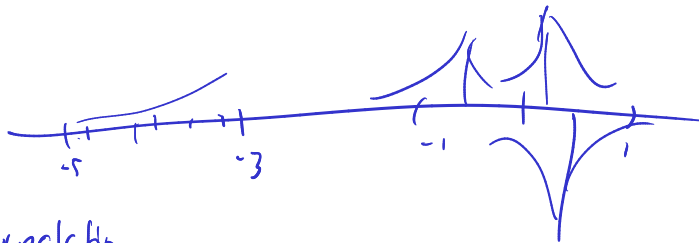


$$\Delta u = 0$$

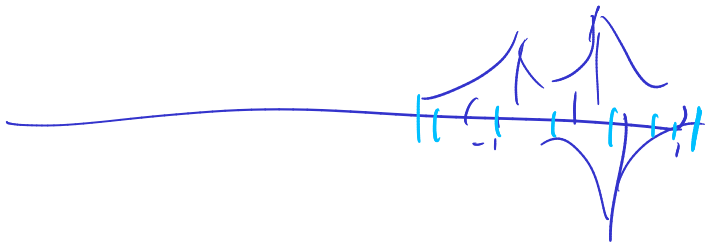
$$\Delta u + k^2 u = 0$$



Interpolation:



Anterpolation



Anterpolation

WLOG: 1D. Assume K smooth, $(y_j)_{j=1}^N \subset [-1, 1]$. Assume N large. Let $u(x) = \sum_{j=1}^N \rho_j K(x - y_j)$. Use interpolation to compress?



How about that smoothness assumption?



Demo: Anterpolation

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Barnes-Hut

Fast Multipole

Ewald Summation and DMK

Direct Solvers

The Butterfly Factorization

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Preliminaries: Convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(\xi)g(x - \xi)d\xi.$$

- Convolution with a shifted δ is the same as shifting the function:

$$[f * (\xi \mapsto \delta(\xi - a))](x) = f(x - a).$$

- Convolution is linear (in both arguments) and commutative.

Preliminaries: Fourier Transform

$$\mathcal{F}(f)(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} dx.$$

$$\mathcal{F}(f)(\omega) = O(|\omega|^{-\alpha}) \\ |\omega| \rightarrow \infty$$

- Convolution turns into multiplication: $\mathcal{F}\{f * g\} = \mathcal{F}f \cdot \mathcal{F}g$.
- A single shifted δ transforms as (as a distribution!)

$$\mathcal{F}\{\delta(x - a)\}(\omega) = e^{-2\pi i a \omega}.$$

- A unit-periodic “train” of δ s is its own $\mathcal{F}$ [Décoret '04]:

$$\mathcal{F}\left\{\sum_{\ell \in \mathbb{Z}} \delta(x - \ell)\right\}(\omega) = \sum_{k \in \mathbb{Z}} \delta(\omega - k).$$

What is $\mathcal{F}\{f(x - a)\}$?

$$\mathcal{F}\{f\}(\omega) e^{-2\pi i a \omega}$$

$f(x)$		$\mathcal{F}$
$\delta(x)$		1
$H(x)$		$\frac{1}{i\omega}$
		$\frac{1}{(i\omega)^2}$

Preliminaries: Non-Uniform FFT (Type-1)

Given nonuniform nodes $x_j \in [0, 2\pi)$ and complex strengths c_j , we want

$$F_k = \sum_{j=0}^{N-1} c_j e^{-ikx_j} \quad (-M/2 \leq k < M/2).$$

- ① Convolve with Gaussians onto reg. grid
- ② FFT
- ③ Divide by $\mathcal{F}\{\text{Gaussian}\}$

Preliminaries: Non-Uniform FFT (Type-2)

Given nonuniform nodes $x_j \in [0, 2\pi)$, we want

$$f(x_j) = \sum_{k=-M/2}^{M/2} F(k) e^{ikx_j}$$

- ① $\mathcal{F}$ -deconvolve (i.e. divide)
- ② FFT
- ③ Convolve from reg. grid onto $\{x_j\}$