October 13, 2026

Announcements

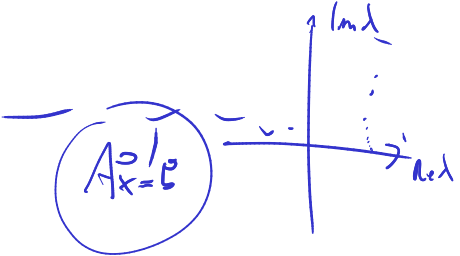
- Project Proposal
- IEF

Goals

- Finish DMK
- Direct solver
- Butterfly

Review

- Ewald
- DMK



Ewald to DMK: Non-adaptive

So what to do?



Algorithmic outline:



Demo: Dual-Space Methods (part I)

Ewald to DMK: Adaptive

How can we make that algorithm adaptive?



Complexity?



Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions
- Barnes-Hut
- Fast Mutipole
- Ewald Summation and DMK
- Direct Solvers
- The Butterfly Factorization

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

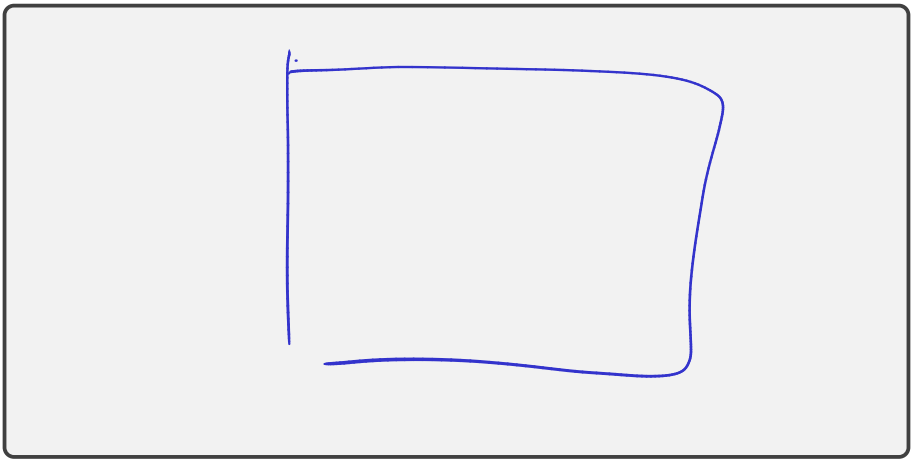
What about solving?

Likely computational goal: Solve a linear system $Ax = b$. How do our methods help with that?

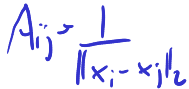
- GMRES
- CG

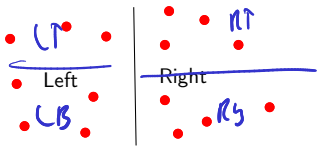
A Matrix View of Low-Rank Interaction

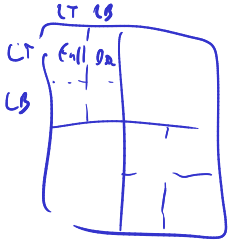
Only *parts of the matrix are low-rank*! What does this look like from a matrix perspective?

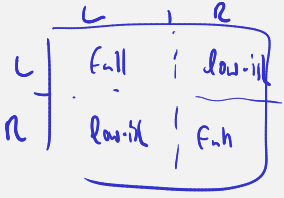


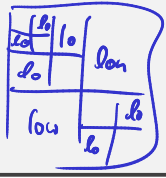
(Recursive) Coordinate Bisection (RCB)





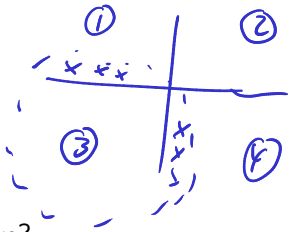




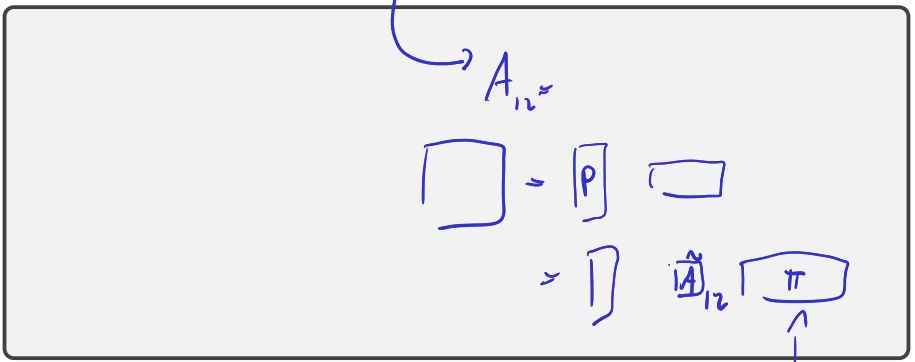


Block-separable matrices

$$A = \begin{bmatrix} D_1 & A_{12} & A_{13} & A_{14} \\ A_{21} & D_2 & A_{23} & A_{24} \\ A_{31} & A_{32} & D_3 & A_{34} \\ A_{41} & A_{42} & A_{43} & D_4 \end{bmatrix}$$



where A_{ij} has low rank: How to capture rank structure?



Proxy Recap

Saw: If A comes from a kernel for which Green's formula holds, then the same skeleton will work for all of space, for a given set of sources/targets. What would the resulting matrix look like?

Rank and Proxies

Unlike FMMs, partitions here do not include "buffer" zones of near elements. What are the consequences?

Block-Separable Matrices

A *block-separable matrix* looks like this:

$$A = \begin{bmatrix} D_1 & P_1\tilde{A}_{12}\Pi_2 & P_1\tilde{A}_{13}\Pi_3 & P_1\tilde{A}_{14}\Pi_4 \\ P_2\tilde{A}_{21}\Pi_1 & D_2 & P_2\tilde{A}_{23}\Pi_3 & P_2\tilde{A}_{24}\Pi_4 \\ P_3\tilde{A}_{31}\Pi_1 & P_3\tilde{A}_{32}\Pi_2 & D_3 & P_3\tilde{A}_{34}\Pi_4 \\ P_4\tilde{A}_{41}\Pi_1 & P_4\tilde{A}_{42}\Pi_2 & P_4\tilde{A}_{43}\Pi_3 & D_4 \end{bmatrix}$$

Here:

- $\tilde{A}_{ij}$ smaller than A_{ij}
- D_i has full rank (not necessarily diagonal)
- P_i shared for entire row
- Π_i shared for entire column

Block-Separable Matrix: Questions

Q: Why is it called that?

Q: How expensive is a matvec?

Q: How about a solve?

BSS Solve (I)

Separate out 'coarse' unknowns. Use the following notation:

$$B = \begin{bmatrix} 0 & P_1\tilde{A}_{12} & P_1\tilde{A}_{13} & P_1\tilde{A}_{14} \\ P_2\tilde{A}_{21} & 0 & P_2\tilde{A}_{23} & P_2\tilde{A}_{24} \\ P_3\tilde{A}_{31} & P_3\tilde{A}_{32} & 0 & P_3\tilde{A}_{34} \\ P_4\tilde{A}_{41} & P_4\tilde{A}_{42} & P_4\tilde{A}_{43} & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & D_4 \end{bmatrix}, \quad \Pi = \begin{bmatrix} \Pi_1 & & & \\ & \Pi_2 & & \\ & & \Pi_3 & \\ & & & \Pi_4 \end{bmatrix}.$$

$A = D + B\Pi$

$$\begin{bmatrix} D & B \\ -\Pi & I \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{\tilde{x}} \end{bmatrix} = \begin{bmatrix} \vec{\zeta} \\ \vec{0} \end{bmatrix} \qquad \Pi\vec{x} = \vec{\tilde{x}}$$