October 20, 2026

Announcements

- Projects continue to iterate
- HW 4

Goals

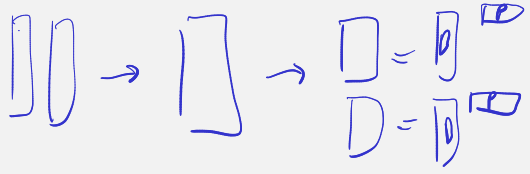
- Butterfly
- ?, Theory
  - PDE solvers
  - Layer potentials
  - Basic functional analysis

Review

- Butterfly

Recompression: Making use of Area-Bounded Rank

How do rectangular submatrices get expressed so as to reveal their constant rank?



Observations

Demo: Butterfly Factorization (Part II)

For which types of matrices is the Butterfly factorization guaranteed accurate?

all of them.

For which types of $n \times n$ matrices does the butterfly lead to a reduction in cost?

"rank tapers off in the middle"

r

Cost

$2^L = O(N)$

What is the cost (in the reduced-cost case) of the matvec?

level 0: $r \times 2r$ matvec $\times 2^L$

⋮

level L: $r \times 2_L$ matvec $\times 2^L$

$O(N \log N)$

Comments?

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

PDEs: Simple Ones First, More Complicated Ones Later

Laplace

$\triangle u = 0$

Applications:

- Steady-state $\partial_t u = 0$ of wave propagation, heat conduction
- Electric potential u for applied voltage
- Minimal surfaces/"soap films"
- ∇u as velocity of incompressible/potential flow

Helmholtz

$\triangle u + k^2 u = 0$

Assume time-harmonic behavior $\tilde{u} = e^{\pm i\omega t} u(x)$ in time-domain wave equation:

$$\partial_t^2 \tilde{u} = \triangle \tilde{u}$$

$\tilde{u}(x,t) = e^{i\omega t} u(x)$

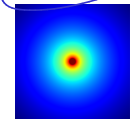
Applications:

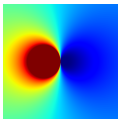
- Propagation of sound
- Electromagnetic waves

Fundamental Solutions

Laplace

$-\triangle u = \delta$





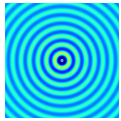
$\varphi \in C^\infty$

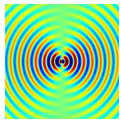
$\int -\Delta u \, \varphi = \int \delta \varphi$

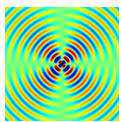
$= \varphi(0)$

Helmholtz

$\triangle u + k^2 u = \delta$





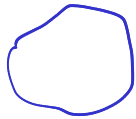


aka. *Free space Green's Functions*

How do you assign a precise meaning to the statement with the δ-function?

mult by smooth test f.; integrate by parts

Green's Functions



Why care about Green's functions?

$u = 0$ on $\partial\Omega$

$$\Delta u = f$$
$$\Delta G = -\delta \quad \Rightarrow \quad \Delta(\underbrace{-G * f}_{u}) = (f * \delta) = f$$

What is a non-free-space Green's function? I.e. one for a specific domain?

$$-\Delta G = \delta \quad + BC$$

$$\int G(x, y) f(y) \, dy$$

Green's Functions (II)

Why not just use domain Green's functions?

math -> too hard.

What if we don't know a Green's function for our PDE... at all?

look at highest order

Fundamental Solutions

Laplace

$$G(x) = \begin{cases} \frac{1}{-2\pi} \log |x| & \text{2D} \\ \frac{1}{4\pi} \frac{1}{|x|} & \text{3D} \end{cases}$$

$$\frac{\partial}{\partial_x} G(x)$$

Helmholtz

$$G(x) = \begin{cases} \frac{i}{4} H_0^1(k|x|) & \text{2D} \\ \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} & \text{3D} \end{cases}$$

$$\frac{\partial}{\partial_x} G(x)$$

Layer Potentials (I)

Let G_k be the Helmholtz kernel ($k = 0 \to$ Laplace).

$\sigma : \Gamma \to \mathbb{C}$

$$(S\sigma)(\vec{x}) = \int_\Gamma G(x-y)\,\sigma(y)\,dS_y \quad \text{"single"}$$

$$(S'\sigma)(\vec{x}) = \partial_{n_x} \int_\Gamma G(x-y)\,\sigma(y)\,dS_y \quad \text{"single"}$$

$$(D\sigma)(\vec{x}) = \int_\Gamma \partial_{n_y} G(x-y)\,\sigma(y)\,dS_y$$

$$(D'\sigma)(\vec{x}) = \partial_{n_y} \int_\Gamma \partial_{n_y} G(x-y)\,\sigma(y)\,dS_y$$

These operators map function σ on Γ to...

$S, D\sigma$ $\mathbb{R}^3 \to \mathbb{C}$ ~~$\mathbb{R}^2 \to \mathbb{C}$~~ ?

$S', D' : \Gamma \to \mathbb{C}$

Layer Potentials (II)

Called *layer potentials*:

- S is called the *single-layer potential*
- D is called the *double-layer potential*
- S'' (and higher) analogously

(Show pictures using `pytential/examples/layerpot.py`, observe continuity properties.)

Alternate ("standard") nomenclature: