October 22, 2026

Announcements

- Projects
- HW4 (due next Thu)

Goals

- LP
  - ↳ PDE solving
- 19th century FA

Review

- Integral equations
  - layer potentials

Layer Potentials (I)

Let G_k be the Helmholtz kernel ($k = 0 \to$ Laplace).

$$(S\sigma)(x) := \int_\Gamma G(x-y)\,\sigma(y)\,dS_y$$

$$(D\sigma)(x) := \int_\Gamma \partial_{n(y)} G(x-y)\,\sigma(y)\,dS_y$$

$$(S'\sigma)(x) := \partial_{n(x)} \int_\Gamma G(x-y)\,\sigma(y)\,dS_y$$

$$(D'\sigma)(x) := \partial_{n(x)} \int_\Gamma \partial_{n(y)} G(x-y)\,\sigma(y)\,dS_y$$

These operators map function σ on Γ to...

$S, D, \sigma : \mathbb{R}^3 \to \mathbb{C} \qquad ?!\ u : \Gamma \to \mathbb{C}$

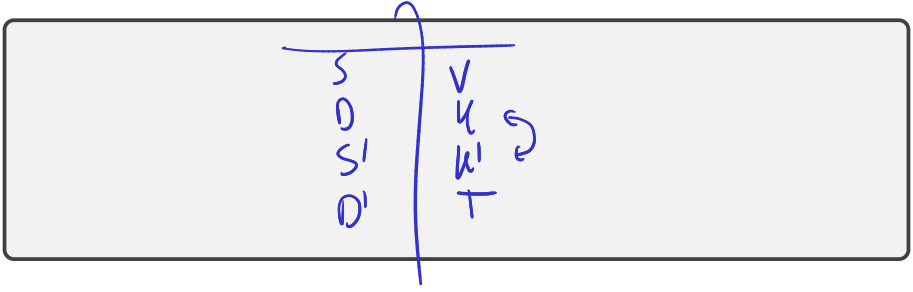
Layer Potentials (II)

Called *layer potentials*:

- S is called the *single-layer potential*
- D is called the *double-layer potential*
- S'' (and higher) analogously

(Show pictures using `pytential/examples/layerpot.py`, observe continuity properties.)

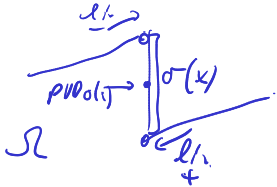
Alternate ("standard") nomenclature:



How does this actually solve a PDE?

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u|_\Gamma = f|_\Gamma.$$



$u(\vec{x}) = (S\sigma)(|\vec{x}|$

$\Delta u(x) = 0$ in Ω ✓

$\lim_{x \to \Gamma^-} u(x) = f$

$S\sigma(x) = f(x)$ on Γ

$\int_\Gamma G(x-y)\,\sigma(y)\,dS_y = f(x)$

"First kind" — BAD

$u(x) = D\sigma(x)$

$\Delta u(x) = 0$ in Ω ✓

$\lim_{x \to \Gamma^-} u(x) = f$

$-\frac{\sigma}{2}(x) + PV\,D\sigma(x) = f(x)$

compact

$(I + PV\,D)\,\sigma(x) = f(x)$

"second-kind"

$$\Delta G(x) = 0$$

$$\Delta \partial_x G(x)$$

$$= \partial_x \underbrace{\Delta G(x)}_{0} = 0$$

IE BVP Solve: Observations (I)

Observations:

- One can choose representations relatively freely. Only constraints:
  - Can I get to the solution with this representation?
    I.e. is the solution I'm looking for represented?
  - Is the resulting integral equation solvable?

  Q: How would we know?

IE BVP Solve: Observations (II)

- Some representations lead to better integral equations than others.
  The one above is actually terrible (both theoretically and practically).
  Fix above: Use $u(x) = D\sigma(x)$ instead of $u(x) = S\sigma(x)$.
  Q: How do you tell a good representation from a bad one?
- Need to actually *evaluate* $S\sigma(x)$ or $D\sigma(x)\ldots$
  Q: How?

$\rightarrow$ Need some theory

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis
- Norms and Operators
- Compactness
- Integral Operators
- Riesz and Fredholm
- A Tiny Bit of Spectral Theory

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

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Norms

Definition

(Norm) A *norm* $\|\cdot\|$ maps an element of a *vector space* into $[0, \infty)$. It satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Can create norm from *inner product*: $\|x\| = \sqrt{\langle x, x \rangle}$

Function Spaces

Name some function spaces with their norms.

L^2

C^0

C^k

H^1

$C^0, \quad \|\cdot\|_\infty$

Convergence

Name some ways in which a sequence can 'converge'.

(x_n) norm conv. $:\Leftrightarrow \quad \|x_n - x\| \to 0$

(x_n) Cauchy $:\Leftrightarrow$

$\forall \varepsilon > 0 \ \exists n : \quad \|x_m - x_\nu\| < \varepsilon$ for $m, \nu \geq n$

conv $\Rightarrow$ Cauchy?

Cauchy $\Rightarrow$ convergent only for Banach space

Operators

X, Y: Banach spaces, $A : X \to Y$ linear operator

$A(x+y) = Ax + Ay$
$\alpha Ax = A\alpha x$

Definition (Operator norm)

$$\|A\| := \sup\{\|Ax\| : x \in X, \|x\| = 1\}$$

Theorem

$\|A\|$ *bounded* $\Leftrightarrow$ A *continuous*

$L(X, Y)$ ← linear bounded operators $X \to Y$ is a Banach space with the operator norm

$\|Ax\| \leq \|A\| \|x\|$
$\|AB\| \leq \|A\| \|B\|$

- What does 'linear' mean here?
- Is there a notion of 'continuous at x' for linear operators?

Operators: Examples

Which of these is bounded as an operator on functions on the real line?

- Multiplication by a scalar
- "Left shift"
- Fourier transform
- Differentiation
- Integration
- Integral operators

Integral Equations: Zoology

	Volterra	Fredholm
First kind	$\int_a^x k(x,y)f(y)dy = g(x)$	$\int_G k(x,y)f(y)dy = g(x)$
Second Kind	$f(x) + \int_a^x k(x,y)f(y)dy = g(x)$	$f(x) + \int_G k(x,y)f(y)dy = g(x)$

Questions:

- First row: First or second kind?
- Second row: Volterra or Fredholm?
- Matrix (i.e. finite-dimensional) analogs?
- What can happen in 2D/3D?
- Factor allowable in front of the identity?
- Why even talk about 'second-kind operators'?
  - Throw a $+\delta(x-y)$ into the kernel, back to looking like first kind. So?
  - Is the identity in $(I+K)$ crucial?

Connections to Complex Variables

Complex analysis is *full* of integral operators:

- Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z-a} f(z)\, dz$$

- Cauchy's differentiation formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{1}{(z-a)^{n+1}} f(z)\, dz$$

Integral Operators: Boundedness (=Continuity)

Theorem (Continuous kernel $\Rightarrow$ bounded)

$G \subset \mathbb{R}^n$ closed, bounded ("compact"), $K \in C(G^2)$. Let

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

Then

$$\|A\|_\infty = \max_{x\in G} \int_G |K(x,y)|dy.$$

Show '$\leqslant$'.

Solving Integral Equations

Given

$$(A\phi)(x) := \int_G K(x,y)\varphi(y)dy,$$

are we allowed to ask for a solution of

$$(\mathrm{Id} - A)\varphi = g?$$

Attempt 1: The Neumann series

Want to solve

$$\varphi - A\varphi = (I - A)\varphi = g.$$

Formally:

$$\varphi = (I - A)^{-1} g.$$

What does that remind you of?

$\frac{1}{1-\alpha} = \sum_{k=0}^{\infty} \alpha^k$

$|\alpha| < 1$

$(I-A)^{-1} = \sum_{k=0} A^k$?

$\|A\| < 1$