October 27, 2026

**Announcements**

- IEF
- HW4
- Project

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**Goals**

- FA land
  - bounded operators
  - $(I-A)\varphi = f$
  - compact op.
  - $\sum$ op. compact
  - Riesz theory

**Review**

$\sum_i G(x_i, y_i)\, \sigma_i$

IMM-like:

- $G(x,y) = \tilde{G}(x-y)$
- smooth / compressible in the far field

What DOES:

| elliptic | parabolic | hyperbolic |
|---|---|---|
| $\Delta u = f$ | yes, but loss relevant on its own | add smoothing |
| $\to$ | | $\to u$ |

# Elliptic

- Volume - homogeneous / exterior /

  $\Delta u = 0$

- Volume - inhom. $\Delta u = f$
- Variable - coeff. $\left(\Delta + k^2(x)\right) u(x) = 0$
  - k piecewise constant
  - general → Lippmann-Schwinger for perturbation




- general method : ?
- exterior : yay!

# Convergence

Name some ways in which a sequence can 'converge'.

$x_n \to x$ if $\|x_n - x\| \to 0$

Cauchy: $\forall \varepsilon > 0 \; \exists n \quad \|x_\mu - x_\nu\| < \varepsilon$ for $\mu, \nu \geq n$

Complete / Banach: Cauchy $\Rightarrow$ convergent.

# Operators

"sequential continuity" $x_n \to x \Rightarrow f(x_n) \to f(x)$

$X, Y$: Banach spaces, $A : X \to Y$ linear operator

**Definition (Operator norm)**

$\|A\| := \sup\{\|Ax\| : x \in X, \|x\| = 1\}$ $\quad \|Ax\| \leq \|A\| \|x\|$

**Theorem**

$\|A\|$ *bounded* $\Leftrightarrow$ *A continuous*

$\|A\| < \infty$

$\mathrm{diag}(1, 2, 3, \ldots)$

- What does 'linear' mean here?
- Is there a notion of 'continuous at $x$' for linear operators?

# Operators: Examples

Which of these is bounded as an operator on functions on the real line?

- Multiplication by a scalar
- "Left shift"
- Fourier transform
- Differentiation
- Integration
- Integral operators

need spaces

# Integral Equations: Zoology

| Volterra | Fredholm |
| --- | --- |
| $\int_a^x k(x,y)f(y)dy = g(x)$ | $\int_G k(x,y)f(y)dy = g(x)$ |
| **First kind** | **Second Kind** |
| $\int_G k(x,y)f(y)dy = g(x)$ | $f(x) + \int_G k(x,y)f(y)dy = g(x)$ |

Questions:

- First row: First or second kind?
- Second row: Volterra or Fredholm?
- Matrix (i.e. finite-dimensional) analogs?
- What can happen in 2D/3D?
- Factor allowable in front of the identity?
- Why even talk about 'second-kind operators'?
  - Throw a $+\delta(x-y)$ into the kernel, back to looking like first kind. So?
  - Is the identity in $(I+K)$ crucial?

need smoothness of k

$$\int_a^x k(x,y) f(y)\,dy = \int_a^b k(x,y)\, 1_{(a,x)}(y)\, f(y)\,dy$$

# Integral Operators: Boundedness (=Continuity)

**Theorem (Continuous kernel ⇒ bounded)**

*$G \subset \mathbb{R}^n$ closed, bounded ("compact"), $K \in C(G^2)$. Let*

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

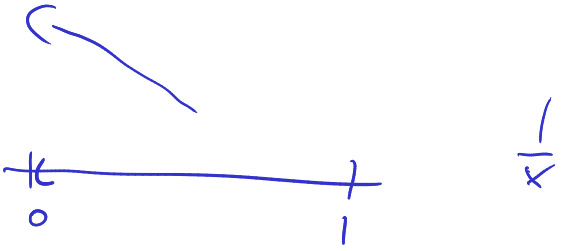
*Then*

$$\|A\|_\infty = \max_{x \in G} \int_G |K(x,y)|dy.$$

Show '$\leqslant$'. Let $\varphi \in C(G)$ with $\|\varphi\|_\infty = 1$.

Let $x \in G$. $\left| \int K(x,y)\varphi(y)\,dy \right| \leq \int |K(x,y)|\,\underbrace{|\varphi(y)|}_{\leq 1}\,dy$

$= \int |K(x,y)|\,dy$


0
1
1/x

# Solving Integral Equations

Given

$$(A\phi)(x) := \int_G K(x,y)\varphi(y)dy,$$

are we allowed to ask for a solution of

$$(\mathrm{Id} - A)\varphi = g?$$

- Neumann series
- Riesz
- Fredholm theory

# Attempt 1: The Neumann series

Want to solve

$$\varphi - A\varphi = (I - A)\varphi = g.$$

Formally:

$$\varphi = (I - A)^{-1} g. \quad \rightsquigarrow \quad \frac{1}{1-\alpha} = \sum_{k=0}^{\infty} \alpha^k$$

What does that remind you of?

# Attempt 1: The Neumann series (II)

**Theorem**

$A : X \to X$ *Banach*, $\|A\| < 1$ $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$ *with*

$\|(I - A)^{-1}\| \leq 1/(1 - \|A\|)$.

- How does this rely on completeness/Banach-ness?
- There's an iterative procedure hidden in this. (Called *Picard Iteration*. Cf: Picard-Lindelöf theorem.) *Hint:* How would you compute $\sum_k A^k f$?

Q: Why does this fall short?

The important cases do not have $\|A\| < 1$.

# Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis
- Norms and Operators
- Compactness
- Integral Operators
- Riesz and Fredholm
- A Tiny Bit of Spectral Theory

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

# Compact Sets

**Definition (Precompact/Relatively compact/Sequentially compact)**

$M \subseteq X$ precompact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in $X$

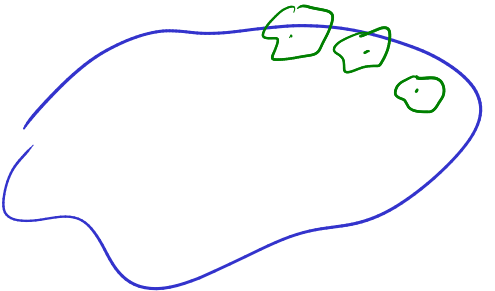
**Definition (Compact/'Sequentially complete')**

$M \subseteq X$ compact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in $M$

- Precompact $\Rightarrow$ bounded
- Precompact $\Leftrightarrow$ bounded (finite dim. only!)
- Question: Is a set consisting of two elements precompact?

**Theorem (Finite subcover)**

*$M \subseteq X$ compact $\Leftrightarrow$ For all covers $C$ with $M = \bigcup_{S \in C} S$ consisting of open sets $S \subseteq X$, there exists a finite subcover $F \subseteq C$ so that $M = \bigcup_{S \in F} S$.*



# Compact Sets (II)

$(x_n)_n$

$\Leftarrow$

Counterexample to 'precompact $\not\Rightarrow$ bounded'? ($\infty$ dim)

look for a bounded set of sequences.

$\|x_n\| \leq 1$

$x_0 = (1, \ldots)$

$x_1 = (\underline{1}, \ldots)$

$\underline{1}$

# Compact Operators

$X, Y$: Banach spaces

$T, S$ bounded

**Definition (Compact operator)**

$T : X \to Y$ is *compact* $:\Leftrightarrow$ $T$(bounded set) is precompact.

**Theorem**

- $T, S$ *compact* $\Rightarrow$ $\alpha T + \beta S$ *compact*
- *One of* $T, S$ *compact* $\Rightarrow$ $S \circ T$ *compact*
- $T_n$ *all compact,* $T_n \to T$ *in operator norm* $\Rightarrow$ $T$ *compact*

Questions:

- Let $\dim T(X) < \infty$. Is $T$ compact?
- Is the identity operator compact?

$(S \circ T)\varphi$

$S(T(\varphi))$