October 29, 2026

Announcements

- HW 4 → due Tue Nov 3
- Project

Goals

- compactness
  (↳ sets
  ↳ operator
  GCIII)
  - $U \subset C(G)$ compact?
    "Arzela-Ascoli"
  - Int. op: compact
  - Riesz.

$(I-A)\varphi = \psi$

Review

Compact Sets

Definition (Precompact/Relatively compact/Sequentially compact)

$M \subseteq X$ precompact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in X

Definition (Compact/'Sequentially complete')

$M \subseteq X$ compact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in M

- Precompact $\Rightarrow$ bounded
- Precompact $\Leftrightarrow$ bounded (finite dim. only!)
- Question: Is a set consisting of two elements precompact?

Theorem (Finite subcover)

$M \subseteq X$ compact $\Leftrightarrow$ For all covers C with $M = \bigcup_{S \in C} S$ consisting of open sets $S \subseteq X$, there exists a finite subcover $F \subseteq C$ so that $M = \bigcup_{S \in F} S$.

Compact Operators

X, Y: Banach spaces. T, S : bounded

Definition (Compact operator)

$T : X \to Y$ is *compact* $:\Leftrightarrow$ T(bounded set) is precompact.

Theorem

- T, S *compact* $\Rightarrow$ $\alpha T + \beta S$ *compact*
- *One of* T, S *compact* $\Rightarrow$ $S \circ T$ *compact*
- T_n *all compact,* $T_n \to T$ *in operator norm* $\Rightarrow$ T *compact*

Questions: $\|T_n - T\| \to 0$

- Let $\dim T(X) < \infty$. Is T compact?
- Is the identity operator compact?

Let $U \subseteq X$ bounded. T, S compact.

Show: $(T+S)(U)$ precompact.

~~Let $x_n \in T(U)$~~

Let $\varphi_n \subset U$. To show: $\big((T+S)(\varphi_n)\big)_n$ has a conv. subseq.

$(T\varphi_n)_n$

$n(k)$ $\quad T\varphi_{n(k)} \to T\varphi \quad (k \to \infty)$

$\psi_k := \varphi_{n(k)}$

$$\zeta \psi_{k(\ell)} \to S\psi \qquad \ell \to \infty$$

$$\left((T+S)\, \varphi_{n(k(\ell))} \right|_{\ell}$$

Intuition about Compact Operators

- Compact operator: As finite-dimensional as you're going to get in infinite dimensions.
- Not clear yet–but they are moral (∞-dim) equivalent of a matrix having *low numerical rank*.
- Are compact operators continuous (=bounded)?
- What do they do to high-frequency data?
- What do they do to low-frequency data?

Arzelà-Ascoli

Let $G \subset \mathbb{R}^n$ be compact.

Theorem (Arzelà Ascoli [Kress LIE 3rd ed. Thm. 1.18])

$U \subset C(G)$ is precompact iff it is bounded and equicontinuous.

Equicontinuous means

For all $\varepsilon > 0$ there exists a $\delta > 0$ s.t. For all $f \in U$,
$\|\vec{x} - \vec{y}\| < \delta \Rightarrow |f(\vec{x}) - f(\vec{y})| < \varepsilon$

Continuous means:

For all $\varepsilon > 0$ there exists a $\delta > 0$ s.t.,
$\|\vec{x} - \vec{y}\| < \delta \Rightarrow |f(\vec{x}) - f(\vec{y})| < \varepsilon$

Arzelà-Ascoli: Proof Sketch for b ∧ e ⇒ precompact

- Let $(f_n)_n \subset U$. Need a subseq. s.t. it's Cauchy.
- Pick a dense sequence $(x_i)_i \subset G$.
- Subseq. of subseq $i(k)$ $x_{i(a)}$

Fix n: $f_n(x_{i(a)})$ is Cauchy.

$\forall \varepsilon > 0, \exists n_0, \forall k, l \geq n_0 \forall x$

$$\left| f_{n(k)}(\vec{x}) - f_{n(l)}(\vec{x}) \right| < \varepsilon$$