September 8, 2026

Announcements

- HW2

Goals

- SVD of low-rank A in $O(\underset{\sim}{N} k^2)$
- smoothness
  → expansions

Review

- Interpolative decomp.
  ↳ CRA

$Q^T A$

Range finder: suppose we know rank k

$\Omega \in \mathbb{R}^{N \times (k+p)}$ $\qquad A \in \mathbb{R}^{N \times N}$

$Y = A\Omega$ $\qquad$ (hopefully: via $O(N)$ matvec)

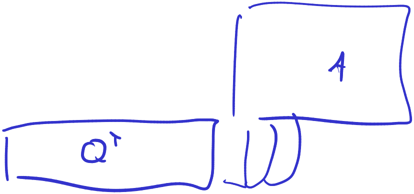
$\hookrightarrow Y = (AA^T)^q A \Omega$

$Q\cancel{R} = Y$ $\quad N \times (k+p)$

$\to Q$

$A \approx QQ^T A$

$Q^T A$

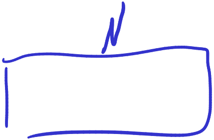


Interpolative decomp.

$$z \approx P \, z_{[J,:]}$$

Instead of $Q^T A$:

$A_{[J,:]} \rightsquigarrow$ keep



ID Q vs ID A

$A \approx QQ^T A$

What does row selection mean for the LRA?

$Q \approx P Q_{[J,:]}$

$J \in \{1, \dots, N\}^{k+p}$

$A \approx P Q_{[J,:]} Q^T A$

$A_{[J,:]} \approx \underbrace{\cancel{P_{[J,:]}}}_{\mathrm{Id}} Q_{[J,:]} Q^T A$

$P A_{[J,:]}$

[Martinsson, Rokhlin, Tygert '06]

Demo: Reusing Decompressors

ID: Remarks

Slight tradeoff here: what?

How would we use the ID in the context of the range finder?

Demo: Interpolative Decomposition

Name a property that the ID has over other factorizations.

ID: Impact on Low-Rank Algorithms

All our randomized tools have two stages:

1. Cast columns of A in terms of 'small' coordinates with respect to a bigger basis.
2. Do actual work only on coordinate representation

Complexity?

What is the impact of the ID?

Leveraging the ID for SVD (I)

Build a low-rank SVD with row extraction.

$$\underset{N\times N}{A} = \underset{N\times k}{P}\ \underset{k\times N}{A_{[J]}}$$

① Compute a row QR:

$$\underbrace{\left(A_{[J]}\right)^T}_{N\times k} = \underset{N\times k}{\bar{Q}}\ \underset{k\times}{\bar{R}}$$

② Upsample row coefficients $\bar{R}^T$

$$\underset{N\times k}{Z} = \underset{N\times k}{P}\ \underset{k\times k}{\bar{R}^T}$$

③

$$\underset{N\times k}{U}\ \underset{k\times k}{\Sigma}\ \underset{k\times k}{\tilde{V}^T} = \underset{N\times k}{Z}$$

Leveraging the ID for SVD (II)

In what way does this give us an SVD of A?

$$
\begin{aligned}
& U \Sigma (\bar{Q} \tilde{V})^T \\
&= U \Sigma \tilde{V}^T \bar{Q}^T \\
&= Z \bar{Q}^T \\
&= P \bar{R}^T \bar{Q}^T \\
&= P A_{[J]}
\end{aligned}
$$

Leveraging the ID for SVD (III)

Q: Why did we need to do the row QR?

Where are we now?

- We have observed that we can make matvecs faster if the matrix has low-ish numerical rank
- In particular, it seems as though if a matrix has low rank, there is no end to the shenanigans we can play.
- We have observed that some matrices we are interested in (in some cases) have low numerical rank (cf. the point potential example)
- We have developed a set of tools that lets us obtain LRAs and do useful work (using SVD as a proxy for "useful work") in $O(N \cdot k^{\alpha})$ time (assuming availability of a cheap matvec).

Next stop: Get some insight into *why* these matrices have low rank in the first place, to perhaps help improve our machinery even further.

Outline

Outline

Punchline

What do (numerical) rank and smoothness have to do with each other?

- Polys
- sin / cos
- eigenfunctions of Sturm-Liouville
- . . .

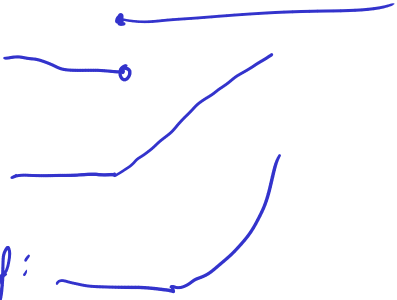
Even shorter punchline?

$$D^{-1}f(x) := \int_a^x f(y)\,dy$$

$f:$

$D^{-1}f:$

$(D^{-1})^2 f:$



Smoothing Operators

If the operations you are considering are *smoothing*, you can expect to get a lot of mileage out of low-rank machinery.

What types of operations are smoothing?

$\int$

Now: Consider some examples of smoothness, with justification.

How do we judge smoothness?

Taylor

Recap: Multivariate Taylor

$$f(c+h) \approx \sum_{p=0}^{k} \frac{f^{(k)}(c)}{k!} h^p \qquad (1D)$$

Multi-index : n dim

$$\vec{p} \in \mathbb{N}_0^n \qquad \vec{p} = (0,1,2)$$

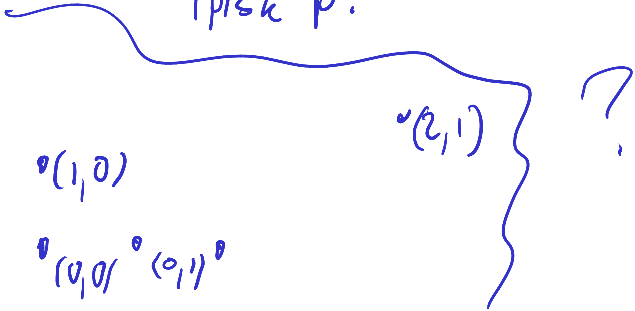
$$|\vec{p}| = p_1 + \dots + p_n$$

$$\vec{p}! = p_1!\, p_2! \dots p_n!$$

$$\vec{x} \in \mathbb{R}^n \qquad \vec{x}^{\vec{p}} = x_1^{p_1} \dots x_n^{p_n}$$

$$D^{\vec{p}} F = \frac{\partial^{|p|}}{\partial x_1^{p_1} \dots \partial x_n^{p_n}}$$

$$f(\vec{c}+\vec{h}) \approx \sum_{|\vec{p}|\le k} \frac{D^{\vec{p}} f(\vec{c})}{\vec{p}\,!}\, \vec{h}^{\vec{p}}$$



$$\left| f(\vec{c}+\vec{h}) - \sum_{\vec{p}} \frac{D^{\vec{p}} f(\vec{c})}{\vec{p}\,!}\, \vec{h}^{\vec{p}} \right| = O(\|h\|^{k+1})$$

$|\vec{p}| \leq k$

P^k

$\max p_i < k$

Q^k