September 15, 2026

Announcements

Goals

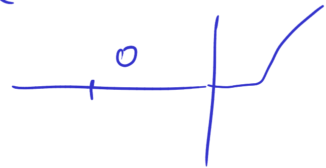
- cliff hanger
- local
- multipole
- others

expansions

Review

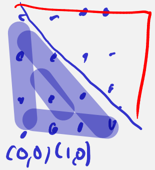
- Smoothness → low rank
- Expansions

C^∞



Recap: Multivariate Taylor

1D Taylor $f(c+h) \approx \sum_{p=0}^{k} \frac{f^{(p)}(c)}{p!} h^p$



Q^N : Full

P^N : $|\vec{p}| \le k$

$p \in \mathbb{N}_0^d$

nD Taylor $f(\vec{c}+\vec{h}) \approx \sum_{|\vec{p}| \le k} \frac{\partial^{\vec{p}}(\vec{c})}{\vec{p}\,!} \vec{h}^{\vec{p}}$

$\vec{p} = (1,0,2)$

Taylor and Error (I)

How can we estimate the error in a Taylor expansion?

$$\left| f(c+h) - \sum_{p=0}^{\infty} \frac{f^{(p)}(c)}{p!} h^p \right|$$

$$= \left| \sum_{p=k+1}^{\infty} \frac{f^{(p)}(c)}{p!} h^p \right|$$

Taylor and Error (II)

Now suppose that we had an estimate that $0 \le \left|\frac{f^{(p)}(c)}{p!}h^p\right| \leqslant \alpha^p.$

$$\left|\sum_{p=k+1}^{\infty} \frac{f^{(p)}(c)}{p!} h^p\right|$$

$$\le \sum_{p=k+1}^{\infty} \left|\frac{f^{(p)}(c)}{p!} h^p\right| \le \sum_{p=k+1}^{\infty} \alpha^p = \frac{1}{1-\alpha}\alpha^{k+1}$$

$$|\alpha| \le 1$$

Connect Taylor and Low Rank

Can Taylor help us establish low rank of an interaction?

$$f(\vec{x}) = f(\vec{c}+\vec{h}) = \sum_{|\vec{p}|\le h} \frac{D^{\vec{p}} f(\vec{c})}{\vec{p}!} \vec{h}^{\vec{p}}$$

$$= \sum_{|\vec{p}|\le h} (\text{coeff})_{\vec{p}} \cdot \vec{h}^{\vec{p}}$$

$$= \sum_{|\vec{p}|\le h} (\text{coeff})_{\vec{p}} \cdot h_1^{p_1} \cdots h_d^{p_d}$$

Taylor on Potentials (I)

Compute a Taylor expansion of a 2D Laplace point potential.

$$\psi(\vec{x}) = \sum_{j=1}^{n} G(\vec{x} - \vec{y}_j)\,\varphi(\vec{y}_j)$$

$$= \sum_{j=1}^{n} \log(\|\vec{x} - \vec{y}_j\|)\,\varphi(\vec{y}_j)$$

$$\frac{D^p \psi}{\vec{p}!}$$

Taylor on Potentials (Ia)

Why is it interesting to consider Taylor expansions of Laplace point potentials?

Taylor on Potentials (II)

```
Maxima 5.42.1 http://maxima.sourceforge.net
(%i1) phi0: log(sqrt(y1**2 + y2**2));
                                       2     2
                                 log(y2  + y1 )
(%o1)                            --------------
                                       2
(%i2) diff(phi0, y1);
                                       y1
(%o2)                               ---------
                                      2     2
                                    y2  + y1
(%i3) diff(phi0, y1, 5);
                                          3              5
                120 y1          480 y1          384 y1
(%o3)         ------------ - ------------ + ------------
                2      2 3      2      2 4      2      2 5
              (y2  + y1 )    (y2  + y1 )    (y2  + y1 )
(%i4)
```

Taylor on Potentials (III)

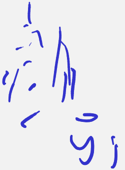
Which of these is the most dangerous (largest) term? Let $R = \sqrt{y_1^2 + y_2^2}$. What's a bound on it? (as $R \to 0$)

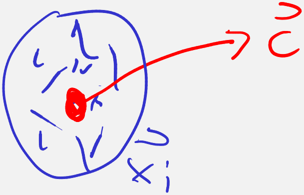
'Generalize' this bound:

$$\left| \frac{D^{\vec{p}} \psi}{p!} \right| \le C \begin{cases} \log R & |\vec{p}| = 0 \\ R^{-|\vec{p}|} & \text{otherwise} \end{cases}$$

Taylor on Potentials (IV)

What does this mean for the convergence of the Taylor series as a whole?





"local expansion"

$$\left| \frac{D_z^{\vec{p}} \psi(\vec{z})\big|_{z=\vec{c}-\vec{y}}}{\vec{p}!} \vec{h}^{\vec{p}} \right| \le C \left(\frac{\|h\|_2}{R} \right)^{|\vec{p}|}$$

Taylor on Potentials (V)

Lesson?

$$\frac{\max_i \|x_i - c\|_2}{\min \|y_j - c\|_2} = \frac{r}{R} < 1$$

Taylor on Potentials (VI)

Generalize this to multiple source points:

Local expansions as a Computational Tool

Low rank makes evaluating interactions cheap(er). Do local expansions help with that goal?

Outline

Taylor on Potentials, Again

Stare at that Taylor formula again. ⚠ sep. of variables

$$\psi(\vec{x}-\vec{y}) \approx \sum_{|\vec{p}|\le k} \underbrace{\frac{D^{\vec{p}}\psi(z)|_{z=c-y}}{\vec{p}!}}_{\text{dep. on src/ctr coeff}} \underbrace{(x-c)^{\vec{p}}}_{\text{dep. on tgt/ctr basis}}$$

$$\psi(\vec{x}-\vec{y}) \approx \sum_{|\vec{p}|\le k} \underbrace{\frac{(-1)^{|\vec{p}|} D^{\vec{p}}\psi(z)|_{z=\vec{x}-\vec{c}}}{\vec{p}!}}_{\text{dep. on tgt/ctr basis}} \underbrace{(\vec{y}-\vec{c})^{\vec{p}}}_{\text{dep. on src/ctr coeff.}}$$

Multipole Expansions (I)

At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

First Q: When does this expansion converge?

$$\frac{\max_j \|\vec{y}_j - \vec{c}\|_2}{\min_i \|x_i - \vec{c}\|_2}$$