

September 15, 2026

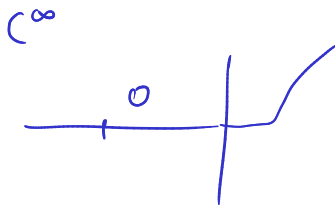
Announcements

Goals

- cliff hanger
 - local
 - multipole
 - others
- } expanding

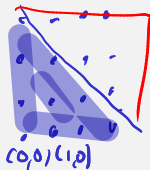
Review

- Smoothness $\rightarrow$ low rank
- Expansions



Recap: Multivariate Taylor

1D Taylor $f(c+h) \approx \sum_{p=0}^k \frac{f^{(p)}(c)}{p!} h^p$



Q^N : total
 $P^N : |\vec{p}| \leq k$

nD Taylor $f(\vec{c}+\vec{h}) \approx \sum_{|\vec{p}| \leq k} \frac{\partial^{\vec{p}} f(\vec{c})}{\vec{p}!} \vec{h}^{\vec{p}}$

$p \in \mathbb{N}_0^d$
 $\vec{p} = (1, 0, 2)$

Taylor and Error (I)

How can we estimate the error in a Taylor expansion?

$$\left| f(c+h) - \sum_{p=0}^{\infty} \frac{f^{(p)}(c)}{p!} h^p \right|$$
$$= \left| \sum_{p=k+1}^{\infty} \frac{f^{(p)}(c)}{p!} h^p \right|$$

Taylor and Error (II)

Now suppose that we had an estimate that

$$\left| \frac{f^{(p)}(c)}{p!} h^p \right| \leq \alpha^p.$$

$$\frac{1}{1-\alpha}$$

$$\left| \sum_{p=k+1}^{\infty} \frac{f^{(p)}(c)}{p!} h^p \right|$$

$$\leq \sum_{p=k+1}^{\infty} \left| \frac{f^{(p)}(c)}{p!} h^p \right| \leq \sum_{p=k+1}^{\infty} \alpha^p = \frac{1}{1-\alpha} \alpha^{k+1}$$

$$|\alpha| < 1$$

Connect Taylor and Low Rank

Can Taylor help us establish low rank of an interaction?

$$f(\vec{x}) = f(\vec{c} + \vec{t}) = \sum_{|\vec{p}| \leq h} \frac{D^{\vec{p}} f(\vec{c})}{p!} \vec{h}^{\vec{p}}$$

$$= \sum_{|\vec{p}| \leq h} (\text{coeff})_{\vec{p}} \cdot \vec{h}^{\vec{p}}$$

$$= \sum_{|\vec{p}| \leq h} (\text{coeff})_{\vec{p}} \cdot h_1^{p_1} \cdots h_d^{p_d}$$

Taylor on Potentials (I)

Compute a Taylor expansion of a 2D Laplace point potential.

$$\begin{aligned}\psi(\vec{x}) &= \sum_{j=1}^n G(\vec{x} - \vec{y}_j) \varphi(\vec{y}_j) \\ &= \sum_{j=1}^n \log(\|\vec{x} - \vec{y}_j\|) \varphi(\vec{y}_j)\end{aligned}$$

$$\frac{D^p \psi}{p!}$$

Taylor on Potentials (Ia)

Why is it interesting to consider Taylor expansions of Laplace point potentials?



Taylor on Potentials (II)

Maxima 5.42.1 <http://maxima.sourceforge.net>

(%i1) phi0: log(sqrt(y1**2 + y2**2));

(%o1)

$$\frac{\log(y_2^2 + y_1^2)}{2}$$

(%i2) diff(phi0, y1);

(%o2)

$$\frac{y_1}{y_2^2 + y_1^2}$$

(%i3) diff(phi0, y1, 5);

(%o3)

$$\frac{120 y_1}{(y_2^2 + y_1^2)^3} - \frac{480 y_1}{(y_2^2 + y_1^2)^4} + \frac{384 y_1}{(y_2^2 + y_1^2)^5}$$

(%i4)

Taylor on Potentials (III)

Which of these is the most dangerous (largest) term? Let $R = \sqrt{y_1^2 + y_2^2}$.
What's a bound on it? (as $R \rightarrow 0$)

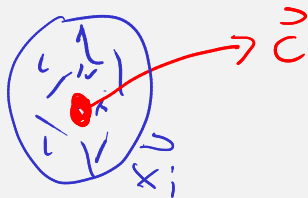


'Generalize' this bound:

$$\left| \frac{D^{\vec{p}} \psi}{p!} \right| \leq C \begin{cases} \log R & |\vec{p}|=0 \\ R^{-|\vec{p}|} & \text{otherwise} \end{cases}$$

Taylor on Potentials (IV)

What does this mean for the convergence of the Taylor series as a whole?



"local expansion"

$$\left| \frac{D_{\vec{z}}^{\vec{p}} \psi(\vec{z})}{\vec{p}!} \right|_{\vec{z} = \vec{c} - \vec{y}} \leq C \left(\frac{\|h\|_2}{R} \right)^{|\vec{p}|}$$

Taylor on Potentials (V)

Lesson?

$$\frac{\max_i \|x_i - c\|_2}{\min_j \|\tilde{y}_j - c\|_2} = \frac{r}{R} < 1$$

Taylor on Potentials (VI)

Generalize this to multiple source points:



Local expansions as a Computational Tool

Low rank makes evaluating interactions cheap(er). Do local expansions help with that goal?



Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Local Expansions

Multipole Expansions

Rank Estimates

Proxy Expansions

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Taylor on Potentials, Again

Stare at that Taylor formula again.

Δ sep. of variables

$$\psi(\vec{x} - \vec{y}) \approx \sum_{|\vec{p}| \leq k} \underbrace{\frac{D^{\vec{p}} \psi(z)|_{z=\vec{c}-\vec{y}}}{\vec{p}!}}_{\text{dep. on src/ctr coeff}} \underbrace{(\vec{x} - \vec{c})^{\vec{p}}}_{\text{dep. on tgt/ctr basis}}$$

$$\psi(\vec{x} - \vec{y}) \approx \sum_{|\vec{p}| \leq k} \underbrace{\frac{(-1)^{|\vec{p}|} D^{\vec{p}} \psi(z)|_{z=\vec{x}-\vec{c}}}{\vec{p}!}}_{\text{dep. on tgt/ctr basis}} \underbrace{(\vec{y} - \vec{c})^{\vec{p}}}_{\text{dep. on src/ctr coeff.}}$$

Multipole Expansions (I)

At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

First Q: When does this expansion converge?

$$\frac{\max_j \| \vec{y}_j - \vec{c} \|_2}{\min_i \| x_i - \vec{c} \|_2}$$

