

September 17, 2026

Announcements

- HW2 : due next week

Goals

- demo
- Taylor $\rightarrow$ rank?
- Alternatives to Taylor
- "Expansion" via linear algebra

Review

Sources
targets
center

- Local and multipole
- Convergence theory

Local	Multipole
	
$Err \sim \left(\frac{d(c, \text{farthest}_{tgt})}{d(c, \text{closest}_{src})} \right)^{p+1}$	$Err \sim \left(\frac{d(c, \text{farth.}_{src})}{d(c, \text{closest}_{tgt})} \right)^{p+1}$

Taylor on Potentials, Again

Stare at that Taylor formula again.

$$(la) \quad \psi(\vec{x} - \vec{y}) \approx \sum_{|\vec{p}| \leq k} \frac{D^{\vec{p}} \psi(\vec{z})|_{\vec{z} = \vec{c}}}{\vec{p}!} \underbrace{(\vec{x} - \vec{c})^{\vec{p}}}_{\text{basis}}$$

$$(impole) \quad \psi(\vec{x} - \vec{y}) \approx \sum_{|\vec{p}| \leq k} \frac{(-1)^{|\vec{p}|} D^{\vec{p}} \psi(\vec{z})|_{\vec{z} = \vec{x} - \vec{c}}}{\vec{p}!} \underbrace{(\vec{y} - \vec{c})^{\vec{p}}}_{\text{basis}}$$

Multipole Expansions (I)

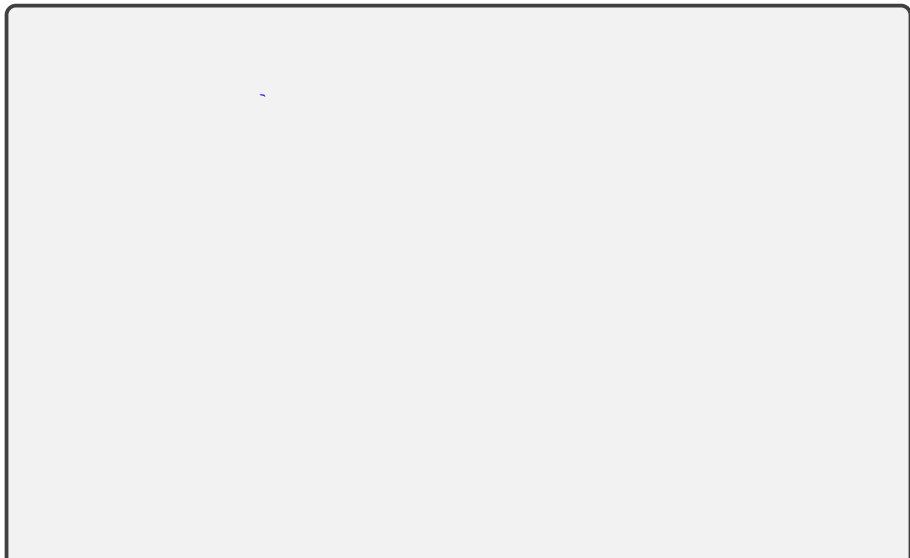
At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

First Q: When does this expansion converge?

Gauger's:
center $\begin{pmatrix} z_1 \text{ nfgts} \\ z_1 : \end{pmatrix}$

Multipole Expansions (II)

The abstract idea of a *multipole expansion* is that:



Multipole Expansions (III)

If our particle distribution is like in the figure, is a multipole expansion computationally useful?

Set

- ▶ $S = \# \text{sources}$,
- ▶ $T = \# \text{targets}$,
- ▶ $K = \# \text{terms in expansion}$.



Demo: Multipole/local expansions

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Local Expansions

Multipole Expansions

Rank Estimates

Proxy Expansions

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Taylor on Potentials: Low Rank?

Connect this to the numerical rank observations:

$$\mathcal{M}(k) = \{ \vec{p} : |\vec{p}| \leq k \}$$

$$2D: \quad |\mathcal{M}(k)| = \frac{(k+1)(k+2)}{2} = O(k^2)$$

$$3D: \quad |\mathcal{M}(k)| = \frac{(k+1)(k+2)(k+3)}{2 \cdot 3} = O(k^3)$$

On Rank Estimates

So how many terms do we need for a given precision ϵ ?

$$\epsilon \approx \left(\frac{d(c, \text{further tgt})}{d(c, \text{closest src.})} \right)^{k+1} = g^{k+1}$$
$$K \approx k^2 \quad \Leftrightarrow \quad k \approx \sqrt{K}, \text{ so } \epsilon \approx g^{\sqrt{K}+1}$$

$$\log \epsilon \approx (\sqrt{K}+1) / \log g$$

$$\sqrt{K}+1 \approx \frac{\log \epsilon}{\log g}$$

$$K \approx \left(\frac{\log \epsilon}{\log g} - 1 \right)^2$$

Demo: Checking rank estimates

Estimated vs Actual Rank

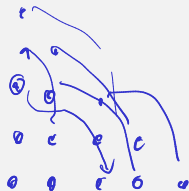
Our rank estimate was off by a power of $\log \varepsilon$. What gives?



Taylor and PDEs

$$\partial_x^2 = -\partial_y^2$$

Look at $\partial_x^2 G$ and $\partial_y^2 G$ in the multipole demo again. Notice anything?



$$O(k^2) \rightarrow O(k)$$

$$\Delta u = 0$$
$$u = g$$

Being Clever about Expansions

How could one be clever about expansions? (i.e. give examples)

$$\Delta u(x,y) = 0 \Leftrightarrow u(x+iy) = u(x,y)$$

↳ do Taylor in complex variables



- Turn PDE into an ODE in $R(r) \cdot e^{im\varphi}$

↳ ODE for $R(r)$ for each m

Expansions for Helmholtz

How do expansions for other PDEs arise?



DLMF 10.23.6 shows 'Graf's addition theorem':

$$H_0^{(1)}(\kappa \|\mathbf{x} - \mathbf{y}\|_2) = \sum_{\ell=-\infty}^{\infty} \underbrace{H_\ell^{(1)}(\kappa \|\mathbf{y} - \mathbf{c}\|_2) e^{i\ell\theta'}}_{\text{singular}} \underbrace{J_\ell(\kappa \|\mathbf{x} - \mathbf{c}\|_2) e^{-i\ell\theta}}_{\text{nonsingular}}$$

where $\theta = \angle(\mathbf{x} - \mathbf{c})$ and $\theta' = \angle(\mathbf{y} - \mathbf{c})$.

Can apply same family of tricks as with Taylor to derive multipole/local expansions.