September 17, 2026

Announcements

- HW2: due next week.

Goals

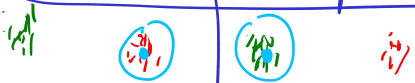
- demo
- Taylor → rank?
- Alternatives to Taylor
- "Expansions" via linear algebra

Review

sources
targets
center

- Local and multipole
- Convergence theory

Local	Multipole



Local: Err $\sim \left(\frac{d(c, \text{farthest tgt})}{d(c, \text{closest src.})}\right)^{p+1}$

Multipole: Err $\sim \left(\frac{d(c, \text{furth. src})}{d(c, \text{closest tgt.})}\right)^{p+1}$

Taylor on Potentials, Again

Stare at that Taylor formula again.

(la) $$\psi(\vec{x}-\vec{y}) \approx \sum_{|\vec{p}|\le k} \frac{D^{\vec{p}}\psi(\vec{z})\big|_{\vec{z}=\vec{c}-\vec{y}}}{\vec{p}!} \underbrace{(\vec{x}-\vec{c})^{\vec{p}}}_{\text{basis}}$$

(mpole) $$\psi(\vec{x}-\vec{y}) \approx \sum_{|\vec{p}|\le k} \underbrace{\frac{(-1)^{|\vec{p}|} D^{\vec{p}}\psi(\vec{z})\big|_{\vec{z}=\vec{x}-\vec{c}}}{\vec{p}!}}_{\text{basis}} (\vec{y}-\vec{c})^{\vec{p}}$$

Multipole Expansions (I)

At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

First Q: When does this expansion converge?

targets: (z, weights)

center (z, :)

Multipole Expansions (II)

The abstract idea of a *multipole expansion* is that:

Multipole Expansions (III)

If our particle distribution is like in the figure, is a multipole expansion computationally useful?

Set

- $S = \#$sources,
- $T = \#$targets,
- $K = \#$terms in expansion.

Demo: Multipole/local expansions

Outline

Taylor on Potentials: Low Rank?

Connect this to the numerical rank observations:

$$\mathcal{M}(k) = \{\vec{p} : |\vec{p}| \le k\}$$

2D: $$|\mathcal{M}(k)| = \frac{(k+1)(k+2)}{2} = O(k^2)$$

3D: $$|\mathcal{M}(k)| = \frac{(k+1)(k+2)(k+3)}{2 \cdot 3} = O(k^3)$$

On Rank Estimates

So how many terms do we need for a given precision ε?

$$\varepsilon \approx \left(\frac{d(c, \text{furthest tgt})}{d(c, \text{closest src.})}\right)^{k+1} = \rho^{k+1}$$

$$K \approx k^2 \iff k \approx \sqrt{K}, \text{ so } \varepsilon \approx \rho^{\sqrt{K}+1}$$

$$\log \varepsilon \approx \left(\sqrt{K}+1\right) \log \rho$$

$$\sqrt{K}+1 \approx \frac{\log \varepsilon}{\log \rho}$$

$$K \approx \left(\frac{\log \varepsilon}{\log \rho} - 1\right)^2$$

Demo: Checking rank estimates

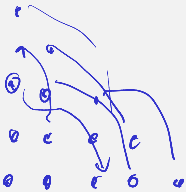
Estimated vs Actual Rank

Our rank estimate was off by a power of $\log \varepsilon$. What gives?

Taylor and PDEs

$$\partial_x^2 = -\partial_y^2$$

Look at $\partial_x^2 G$ and $\partial_y^2 G$ in the multipole demo again. Notice anything?



$$O(k^2) \to O(k)$$

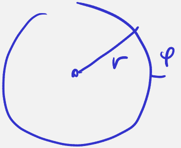
$$\Delta u = 0$$
$$u = y$$

Being Clever about Expansions

How could one be clever about expansions? (i.e. give examples)

$\Delta u(x,y) = 0 \iff \tilde{u}(x+iy) = u(x,y)$

↳ do Taylor in complex variables



- Turn PDE into an ODE in $R(r) \cdot e^{im\varphi}$

↳ ODE for $R(r)$ for each m.

Expansions for Helmholtz

How do expansions for other PDEs arise?

DLMF 10.23.6 shows 'Graf's addition theorem':

$$H_0^{(1)}(\kappa \|\boldsymbol{x}-\boldsymbol{y}\|_2) =$$

$$\sum_{\ell=-\infty}^{\infty} \underbrace{H_\ell^{(1)}(\kappa \|\boldsymbol{y}-\boldsymbol{c}\|_2)\, e^{i\ell\theta'}}_{\text{singular}} \underbrace{J_\ell(\kappa \|\boldsymbol{x}-\boldsymbol{c}\|_2)\, e^{-i\ell\theta}}_{\text{nonsingular}}$$

where $\theta = \angle(\boldsymbol{x}-\boldsymbol{c})$ and $\theta' = \angle(\boldsymbol{y}-\boldsymbol{c})$.

Can apply same family of tricks as with Taylor to derive multipole/local expansions.