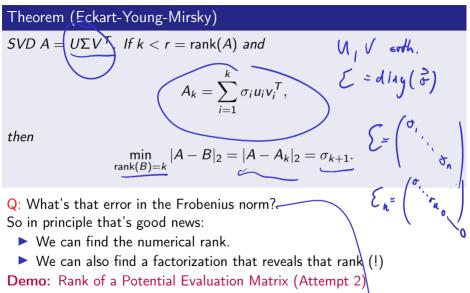




· Vandomized LRA · two different form ;

flwn ABC = (AB)C = A(BC) $AO \times = (AB) \times = A(Ox)$  $(\mathcal{T} \varphi)(x) = \int_{G}^{x} k_{1}(x,y) \varphi(y) dy$ = 56 k, (x,y) 4 (y) dy

# Eckart-Young-Mirsky Theorem



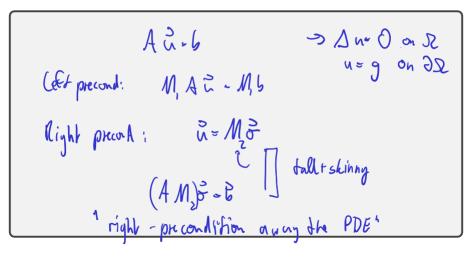
 $\min_{\text{rank}(B)=k} |A - B|_2 = |A - A_k|_2 = \sigma_{k+1}.$  $\min_{\substack{\| A - B \|_{+}}} = \| A - A_{a} \|_{+} =$  $\| \boldsymbol{\xi} - \boldsymbol{\xi}_{\boldsymbol{\mu}} \|_{\boldsymbol{F}}$ 

### Constructing a tool

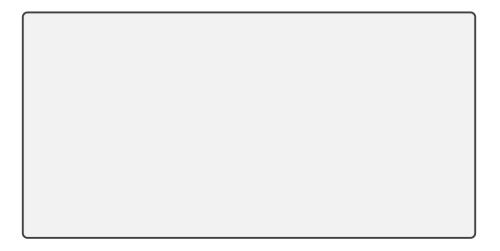
There is still a slight downside, though.

### Representation

What does all this have to do with (right-)preconditioning?



Representation (in context)



# Outline

#### Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra Low-Rank Approximation: Basics Low-Rank Approximation: Error Control Reducing Complexity

#### Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

**Boundary Value Problems** 

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

# Outline

#### Introduction

Dense Matrices and Computation

#### Tools for Low-Rank Linear Algebra Low-Rank Approximation: Basics

Low-Rank Approximation: Basics

Reducing Complexity

### Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

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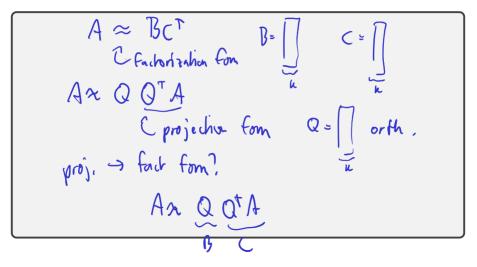
Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

## Rephrasing Low-Rank Approximations

SVD answers low-rank-approximation ('LRA') question. But: too expensive. First, rephrase the LRA problem:



# Using LRA bases

If we have an LRA basis Q, can we compute an SVD?

# Finding an LRA basis

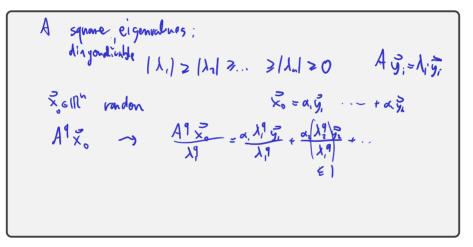
How would we find an LRA basis?

# Giving up optimality

What problem should we actually solve then?

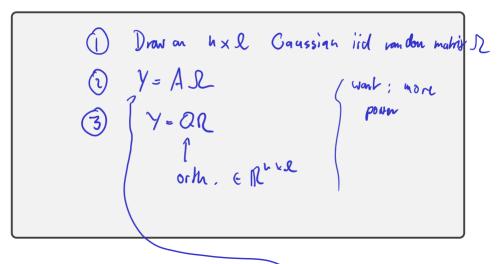
### Recap: The Power Method

How did the power method work again?



### How do we construct the LRA basis?

Put randomness to work:



# Tweaking the Range Finder (I)

Can we accelerate convergence?

$$Y = (A A^{\dagger})^{L} A$$

$$A = U \in V^{T}$$

$$Y = (U \in V^{T} V \in U^{T})^{U} U \in V^{T}$$

$$\sigma_{i} (A A^{\dagger} A) = \sigma_{i} (A)^{2}$$

# Tweaking the Range Finder (II)

What is one possible issue with the power method?

overflow -> normalin 0 · non orth > orthogonalise.

## Even Faster Matvecs for Range Finding

Assumptions on  $\Omega$  are pretty weak-can use more or less anything we want.  $\rightarrow$  Make it so that we can apply the matvec  $A\Omega$  in  $O(n \log l)$  time. How? Pick  $\Omega$  as a carefully-chosen subsampling of the Fourier transform.

# Outline

#### Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra Low-Rank Approximation: Basics Low-Rank Approximation: Error Control Reducing Complexity

#### Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

**Outlook: Building a Fast PDE Solver** 

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

**Boundary Value Problems** 

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

## Errors in Random Approximations

If we use the randomized range finder, how close do we get to the optimal answer?

### Theorem

For an  $m \times n$  matrix A, a target rank  $k \ge 2$  and an oversampling parameter  $p \ge 2$  with  $k + p \le \min(m, n)$ , with probability  $1 - 6 \cdot p^{-p}$ ,

$$|A-QQ^TA|_2 \leq \left(1+11\sqrt{k+p}\sqrt{\min(m,n)}\right)\sigma_{k+1}.$$

(given a few more very mild assumptions on p)

[Halko/Tropp/Martinsson '10, 10.3]

Message: We can probably (!) get away with oversampling parameters as small as p = 5.

### A-posteriori and Adaptivity

The result on the previous slide was *a-priori*. Once we're done, can we find out 'how well it turned out'?

$$E = A - Q Q^{T} A$$

$$||E||_{2} = \delta_{1} (E)$$

$$\tilde{\omega} \in \mathbb{R}^{n} \quad \text{id Gaussian}$$

$$||E||_{2} \approx \max \frac{||E|_{2}}{||\tilde{\omega}||_{2}}$$

# Adaptive Range Finding: Algorithm

# Outline

#### Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra Low-Rank Approximation: Basics Low-Rank Approximation: Error Control Reducing Complexity

#### Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

**Boundary Value Problems** 

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs