

## Announcements

FWI

SFFT → see forum follow-up

## Goals:

- Approx SVD in  $O(N)$  time
- RQRQR
- Interpolative decomp.

## Review:

- $\sqrt{R}$  range Finder  
↳ adaptive
- reconstructing the SVD

$$U \Sigma V^T = \underbrace{Q^T A}$$

Computing  $Q^T A$   $^{k \times N}$  is  $O(N^2 k)$

## Errors in Random Approximations

If we use the randomized range finder, how close do we get to the optimal answer?

### Theorem

*For an  $m \times n$  matrix  $A$ , a target rank  $k \geq 2$  and an oversampling parameter  $p \geq 2$  with  $k + p \leq \min(m, n)$ , with probability  $1 - 6 \cdot p^{-p}$ ,*

$$\|A - QQ^T A\|_2 \leq \left(1 + 11\sqrt{k + p}\sqrt{\min(m, n)}\right) \sigma_{k+1}.$$

*(given a few more very mild assumptions on  $p$ )*

[Halko/Tropp/Martinsson '10, 10.3]

**Message:** We can *probably* (!) get away with oversampling parameters as small as  $p = 5$ .

## A-posteriori and Adaptivity

The result on the previous slide was *a-priori*. Once we're done, can we find out 'how well it turned out'?

$$\|E\| \quad \|A - \alpha \alpha^T A\|$$

$$\|E\|_2 \approx$$

$$\frac{\|A \tilde{w}\|_2}{\|\tilde{w}\|_2}$$

$w_{eff}$  Gaussian iid

## Adaptive Range Finding: Algorithm

(see above)

# Outline

Introduction

Dense Matrices and Computation

**Tools for Low-Rank Linear Algebra**

Low-Rank Approximation: Basics

Low-Rank Approximation: Error Control

**Reducing Complexity**

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

# Rank-revealing/pivoted QR

$$PA = LU$$

Sometimes the SVD is too *good* (aka expensive)—we may need less accuracy/weaker promises, for a significant decrease in cost.

$$A\Pi = QR = Q \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

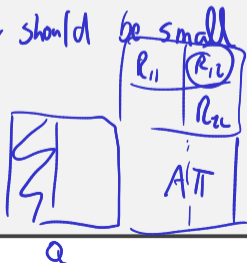
$$R_{11} \in \mathbb{R}^{k \times k}$$

$\|R_{22}\|_2$  hopefully small

$\|R_{12}\|$ ? no reason that should be small

Q orthogonal

$\Pi$  is a permutation



## Using RRQR for LRA

- $\sigma_{k+1} \approx \|R_{22}\|_2$
- To precision,  $\|R_{22}\|_2$ ,  $A$  has at most num. rank  $k$   
(Golub/van Loan, ch.5)

Also suitable for LRA, but:

need heuristic to stop half-way

Does not qualify if full matrices are required.

## Interpolative Decomposition (ID): Definition

Would be helpful to know *columns of A* that contribute 'the most' to the rank.

(orthogonal transformation like in QR 'muddies the waters')

If  $A$  has rank  $k$ , then  $\vec{j}$  is a  $k$ -long index  $v$ .

$$A_{m \times n} = (A_{(:, \vec{j})})_{m \times k} P_{k \times n}$$

- $k$  columns of  $P$  contain a style entry of 1
- " $P$  is well-conditioned"  
all entries are bounded by 2.



## ID: Computation

Assume  $\text{rank}(A) = k$ 

How do we construct this (from RRQR): (short/fat case)

$$A \Pi = Q \begin{bmatrix} R_{11} & R_{12} \\ \hline & \end{bmatrix}$$

$m \times n$   $n \times n$      $m \times k$      $\overbrace{k \times k}^{k \times k}$      $k \times n$

$$B = Q R_{11}$$

$m \times k$      $m \times k$      $k \times k$

Q: What is  $P$ , in terms of the RRQR?

$$P = \begin{bmatrix} I_d & R_{11}^{-1} R_{12} \end{bmatrix} \Pi^T$$

$$BP = Q R_{11} \begin{bmatrix} I_d & R_{11}^{-1} R_{12} \end{bmatrix} \Pi^T$$

$$\rightarrow BP \Pi = Q \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} \Pi^T$$

$$\rightarrow A \Pi = Q \begin{bmatrix} R_{11} & R_{12} \end{bmatrix}$$

# ID $Q$ vs ID $A$

What does row selection mean for the LRA?

$$A \approx Q Q^T A$$

Suppose I run a row ID on  $Q$  :  $Q \approx P Q_{c_j, :}$

$$A_{c_j, :} \approx \overset{\text{Id}}{P_{c_j, :}} Q_{c_j, :} Q^T A$$

[Martinsson, Rokhlin, Tygert '06]

## ID: Remarks

Slight tradeoff here: what?

How would we use the ID in the context of the range finder?

**Demo:** Interpolative Decomposition