Announcements
- HW1

Goals
- Error estimates for Taylor of potentials
- Multipole
- Towards fast algorithms

Review
- \( N^2 k \)
- \( Nk^2 \)
- 1D, cheap LA, SVDs
- Cheap "compressed forms"
- Taylor / smoothness for rank finding
Taylor and Error (II)

Now suppose that we had an estimate that

$$\left| \frac{f^{(p)}(c)}{p!} h^p \right| \leq \alpha^p.$$
Connect Taylor and Low Rank

Can Taylor help us establish low rank of an interaction?

\[ f(c+h) \approx \sum_{\rho=0}^{k} \frac{f^{(\rho)}(c)}{\rho!} h^\rho \]

\[ = \sum_{\rho=0}^{k} \text{coeff}_\rho \text{basis}_\rho(x) \]

\( \Rightarrow \) use this to establish low rank if smooth
Compute a Taylor expansion of a 2D Laplace point potential.

\[ \nabla \psi (\mathbf{x}) = \sum_{j} G(\mathbf{x}, \mathbf{y}_j) \nabla \psi (\mathbf{y}_j) \]

\[ G(\mathbf{x} - \mathbf{y}) = \log \left( \frac{1}{||\mathbf{x} - \mathbf{y}||_2} \right) \]

**Free-space GF:**
\[ A \mathbf{u} = \mathbf{f} \]

**2D Laplace Fundamental Solution:**
\[ \log (x^2 + y^2) \]

**SSE:**
\[ \mathbb{D} \mathbf{D} = \partial_x^2 + \partial_y^2 \]

**WLOG:** consider one source point
\[ \nabla \psi (\mathbf{x}) = \log (||\mathbf{x} - \mathbf{y}||_2) \]

**WLOG:** pick any boundary condition.
\[ \mathcal{Y}(\mathcal{B}) \supseteq \{ \mathcal{P} \} \quad \text{if} \quad |\mathcal{P}| \leq k \]

\[ \frac{\partial^k \mathcal{Y}(\mathcal{B})}{\partial \mathcal{P}^k} \quad \text{for} \quad \mathcal{P}^k \]

Subgoals: Understand growth of those

\[ |\mathcal{P}| = 1 : (0,1) \quad (0,0) \]

\[ |\mathcal{P}| = 2 : (2,0) \quad (1,1) \quad (0,2) \]
\[ \|\mathbf{y}\|_5 \leq C \|\mathbf{y}\|_2 \]

\[
\begin{array}{cccccccc}
10 & 2 & 8 & 4 & 6 \\
24 (y_2 + 5 y_1) & y_2 + 10 y_1 & y_2 + 10 y_1 & y_2 + 5 y_1 & y_2 + y_1
\end{array}
\]

\[
\frac{3 \leq \|\mathbf{y}\|_5}{\|\mathbf{y}\|_2} \sim \frac{1}{\|\mathbf{y}\|_5}
\]
Why is it interesting to consider Taylor expansions of Laplace point potentials?

- pushes boundary of “smooth”
- import and app
Taylor on Potentials (II)

Maxima 5.42.1 http://maxima.sourceforge.net

(%i1) phi0: log(sqrt(y1**2 + y2**2));

\[ \log(y_2 + y_1) \]

(%o1) \frac{2}{2} \log(y_2 + y_1)

(%i2) diff(phi0, y1);

\[ \frac{y_1}{2(y_2 + y_1)} \]

(%o2) \frac{y_1}{2(y_2 + y_1)}

(%i3) diff(phi0, y1, 5);

\[ \frac{120 y_1^3}{2(y_2 + y_1)^3} - \frac{480 y_1^4}{2(y_2 + y_1)^4} + \frac{384 y_1^5}{2(y_2 + y_1)^5} \]

(%o3) \frac{120 y_1^3}{2(y_2 + y_1)^3} - \frac{480 y_1^4}{2(y_2 + y_1)^4} + \frac{384 y_1^5}{2(y_2 + y_1)^5}

(%i4)
Taylor on Potentials (III)

Which of these is the most dangerous (largest) term?

All have the same powers

What’s a bound on it? Let $R = \sqrt{y_1^2 + y_2^2}$. $R = \|y\|_2$

For $p \geq 1$: $\frac{1}{R^p}$

‘Generalize’ this bound:
Taylor on Potentials (IV)

What does this mean for the convergence of the Taylor series as a whole?

\[ | \frac{\partial^p \psi(0)}{p!} h^p | \leq C_p \frac{1}{R^p} | h | h^p = C_p \left( \frac{\| h \|}{R} \right)^p \]

\[ \text{Err est'nhah : } \frac{1}{1 - \alpha} \]

\[ \alpha = \frac{\| h \|}{R} < 1 \]

Conv. crit: \( | \alpha | < 1 \)
Lesson?

\[
\text{Conv. factor: } \frac{1}{\|x - c\|^2} \quad \frac{1}{\|x - c\|^2} \quad \frac{1}{\|y - c\|^2}
\]
Generalize this to multiple source points:

\[
\left( \max_{j} \frac{\|x - y\|_2}{\|y_j - y\|_2} \right)^{p+1} - \left( \frac{\text{dist} (z, \text{farthest target})}{\text{dist} (z, \text{closest source})} \right)^{p+1}
\]
Local expansions as a Computational Tool

Low rank makes evaluating interactions cheap(er). Do local expansions help with that goal?
Find coeffs:

Evaluate expression:

"naive"/direct eval

$O(ST)$

$O(sk) + O(Tk)$

Cost:

$O(sk)$

$L$ forms in the expansion
Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness
  Local Expansions
  Multipole Expansions
  Rank Estimates
  Proxy Expansions

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs
Taylor on Potentials, Again

Stare at that Taylor formula again.

\[ \Psi(x-y) \propto \sum_{|\rho| \leq k} \frac{\partial^\rho \Psi(x-y)}{\rho!} \mid_{x=c} |_{y=c} \]

*(single src, single tgt)*

\[ (x-c)^\rho \]

*dep. on tgt.*

\[ \frac{\partial^\rho \Psi(x-y)}{\rho!} \mid_{y=c} \]

*dep. on src*

\[ (y-c)^\rho \]

*dep. on src*

**Local:**

\[ \Psi(x-y) \propto \sum_{|\rho| \leq k} \frac{\partial^\rho \Psi(x-y)}{\rho!} \mid_{x=c} |_{y=c} (x-c)^\rho \]

*dep. on tgt.*

\[ \frac{\partial^\rho \Psi(x-y)}{\rho!} \mid_{y=c} \]

*dep. on src*

**Multipole expn:**

\[ \Psi(x-y) \propto \sum_{|\rho| \leq k} \frac{\partial^\rho \Psi(x-y)}{\rho!} \mid_{y=c} (y-c)^\rho \]

*dep. on tgt.*

\[ \frac{\partial^\rho \Psi(x-y)}{\rho!} \mid_{y=c} \]

*dep. on src*
Multipole Expansions (I)

At first sight, it doesn’t look like much happened, but mathematically/geometrically, this is a very different animal. **First Q:** When does this expansion converge?