Ann.
- no office hours today + Thu (sorry!)
- project proposals
- HW 4

Goals:
- weakly sing A
  ⇒ compact
- Riesz / Fredholm
- spectral theory

Review

\[(I + A) \psi = \psi\]
compact

Arzela - Ascoli?

\[A \subset \text{linear set.}\]

\[U = \{ \psi \in C^0 : \| \psi \| \infty < \infty \} \]

"bounded"

\[3M > 0 \]

\[U = \{ \psi \in C^0 : \| \psi \| \infty \leq M \} \]

\[A(U) \]
bounded \( \Rightarrow \) A compact
\[ (f')((x) : = \int_0^x f'(y) \, dy = \int_0^x \frac{\delta(x-y)}{\nu(x-y)} \, dy \]

(1) \[ f'((x) : = f'(x) \]

\[ K \text{ continuous } \Rightarrow \int k(x,y) \rho(y) \, dy \text{ compact} \]
Weakly singular

\( G \subset \mathbb{R}^n \) compact

**Definition (Weakly singular kernel)**

- \( K \) defined, continuous everywhere except at \( x = y \)
- There exist \( C > 0, \alpha \in (0, n] \) such that

\[
|K(x, y)| \leq C|x - y|^{\alpha - n} \quad (x \neq y)
\]

**Theorem (Weakly singular kernel \( \Rightarrow \) compact [Kress LIE 3rd ed. Thm. 2.29])**

\( K \) weakly singular. Then

\[
(A\phi)(x) := \int_{G} K(x, y)\phi(y)dy.
\]

is compact on \( C(G) \), where \( \text{cl}(G^\circ) = G \).
Weakly singular: Proof Outline

Outline the proof of ‘Weakly singular kernel ⇒ compact’.

\[ A \varphi(x) = \int \kappa(x,y) \varphi(y) \, dy \]

\[ \int_{B(qd)} \frac{1}{|x-y|^d} \, dy = c_n \quad \int_0^\infty s^{d-n} \rho^{n-1} \, ds = \frac{c_n}{d} \quad \text{bounded} \checkmark \]

\[ h(x) := 1 \quad h \in C^0 \]

\[ K_n(x,y) := h(n|x-y|) \kappa(x,y) \in C^0 \quad K_n \to K \]

\[ (A_n \varphi)(x) = \int k_n(x,y) \varphi(y) \, dy \quad A_n \to A \text{ is now compact} \]
Weakly singular (on surfaces)

\[ \Omega \subset \mathbb{R}^n \text{ bounded, open, } \partial \Omega \text{ is } C^1 \text{ (what does that mean?)} \]

**Definition (Weakly singular kernel (on a surface))**

- \( K \) defined, continuous everywhere except at \( x = y \)
- There exist \( C > 0, \alpha \in (0, n - 1] \) such that

\[
|K(x, y)| \leq C|x - y|^{\alpha - n + 1} \quad (x, y \in \partial \Omega, x \neq y)
\]

**Theorem (Weakly singular kernel \( \Rightarrow \) compact [Kress LIE 3rd ed. Thm. 2.30])**

If \( K \) weakly singular on \( \partial \Omega \). Then \((A\phi)(x) := \int_{\partial \Omega} K(x, y)\phi(y)dy\) is compact on \( C(\partial \Omega) \).

Q: Has this estimate gotten worse or better?
Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

**Going Infinite: Integral Operators and Functional Analysis**
- Norms and Operators
- Compactness
- Integral Operators
  - *Riesz and Fredholm*
  - A Tiny Bit of Spectral Theory

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs
Riesz Theory (I)

Still trying to solve

\[ L\phi := (I - A)\phi = \phi - A\phi = f \]

with \( A \) compact.

Theorem (First Riesz Theorem \([\text{Kress LIE 3rd ed., Thm. 3.1}]\))

\( N(L) \) is finite-dimensional.

Questions:

- What is \( N(L) \) again?
  \[ \text{nullspace}(L) = \{ \phi : \langle \phi, \phi \rangle = 0 \} \]
- Why is this good news?
Riesz First Theorem: Proof Outline

Show it.

\[ \forall \varphi \in N(L) : L \varphi = 0 \quad \Rightarrow \quad (1 - A) \varphi = 0 \quad \Rightarrow \quad \varphi - A \varphi = 0 \]

\[ A|_{N(L)} = I \quad \Rightarrow \quad N(L) \text{ must be } \dim (N(L)) < \infty. \]
Theorem (Riesz theory [Kress LIE 3rd ed., Thm. 3.4])

A compact. Then:

▶ \((I - A)\) injective ⇔ \((I - A)\) surjective

▶ It’s either bijective or neither s nor i.

▶ If \((I - A)\) is bijective, \((I - A)^{-1}\) is bounded.

Rephrase for solvability:

uniqueness ⇒ existence (civilized)

Key shortcoming?

nullspace ⇒ we’re toast.
Riesz Theory: Boundedness Proof Outline

Assuming \((I - A)\) is bijective, show that \((I - A)^{-1}\) is bounded.

Assume \(L^1\) is unbounded. Can pick \(f_n\) with \(\|f_n\| = 1\) with \(\|C f_n\| > n\).

\[ g_n = \frac{f_n}{\|C f_n\|} \to 0 \quad \phi_n = C f_n / \|C f_n\| \]

\[ D \phi_n = \frac{\phi_n}{\|C f_n\|} = C \phi_n = \phi_n - A \phi_n \]

Picks subseq of \(\phi_n\): \(\phi_n(a)\): \(A \phi_n(a) \to \phi\)

(Quite a bit more to the proof: Riesz’ second, third theorems, Riesz numbers, . . . )

\[ A \phi - \phi \Rightarrow \phi \in N(L) \]

Contradicts injectivity.
Hilbert spaces

Hilbert space: Banach space with a norm coming from an inner product:

\[(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)\]

\[(x, \alpha y + \beta z) = \]

\[(x, x) = 0 \iff x = 0\]

\[(y, x) = \]

\[\|x\| = \sqrt{(x, x)}\]

Is \(C^0(G)\) a Hilbert space?

Name a Hilbert space of functions.

\[f, g \in L^2 \Rightarrow f + g \in L^2\]
Can we carry over $C^0(G)$ boundedness/compactness results to $L^2(G)$?

Let $X, Y$ be normed spaces with a scalar product so that $|⟨\phi, \psi⟩| \leq ∥\phi∥∥\psi∥$ for $\phi, \psi \in X$.

**Theorem (Lax dual system [Kress LIE 3rd ed. Thm. 4.13])**

Let $U \subseteq X$ be a subspace and let $A : X \to Y$ and $B : Y \to X$ be bounded linear operators with

$$(A\phi, \psi) = (\phi, B\psi) \quad (\phi \in U, \psi \in Y).$$

Then $A : U \to Y$ is bounded with respect to $∥\cdot∥_s$ induced by the scalar product and $∥A∥^2_s \leq ∥A∥∥B∥$.

Based on this, it is also possible to carry over compactness results.
**Adjoint Operators**

**Definition (Adjoint operator)**

$A^*$ called adjoint to $A$ if

$$(Ax, y) = (x, A^*y)$$

for all $x, y$.

**Facts:**

- $A^*$ unique
- $A^*$ exists
- $A^*$ linear
- $A$ bounded $\Rightarrow$ $A^*$ bounded
- $A$ compact $\Rightarrow$ $A^*$ compact
Adjoint Operator: Observations?

What is the adjoint operator in finite dimensions? (in matrix representation)

What do you expect to happen with integral operators?

Adjoint of the single-layer?

Adjoint of the double-layer?
Fredholm Alternative

Theorem (Fredholm Alternative [Kress LIE 3rd ed. Thm. 4.17])

\[ A : X \to X \text{ compact. Then either:} \]

- \( I - A \) and \( I - A^* \) are bijective
  or:
  - \( \dim N(I - A) = \dim N(I - A^*) \)
  - \( (I - A)(X) = N(I - A^*)^\perp \)
  - \( (I - A^*)(X) = N(I - A)^\perp \)

Seen these statements before?
Fundamental Theorem of Linear Algebra

\[ \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \xleftarrow{A^T} \]

\[ \dim r \quad \begin{cases} 
A^T(\mathbb{R}^m) & \text{dim } r \\
0 & A(\mathbb{R}^n) 
\end{cases} \]

\[ n - r \quad \begin{cases} 
N(A) & m - r \\
0 & N(A^T) 
\end{cases} \]

[Credit: Wikipedia]