Ann:

- no office hours today + Tue (sorry!)
- project proposals
- HW4

Goals:

- weakly sing K $\Rightarrow$ compact
- Riesz / Fredholm
- spectral theory

Review

$(I + A)\varphi = y$

compact

Arzela - Ascoli?

$A \leftarrow$ layer pot.

$\cancel{U = \{\varphi \in C^0 : \|\varphi\| < \infty\}}$ ~~"bounded"~~

$\exists M > 0$

$U = \{\varphi \in C^0 : \|\varphi\|_\infty \leq M\}$

$\cancel{A(U) \subset U?}$

A compact : A(U) bounded compact $\Leftarrow$ A bounded

$\Leftrightarrow$ A cont.

"antiderivative"

$$(\mathcal{I}f)(x) := \int_0^x f(\xi)\,d\xi = \int_0^1 \underbrace{1_{\{\xi < x\}}}_{K(x,\xi)} f(\xi)\,d\xi$$

$$(\mathcal{D}f)(x) := f'(x)$$

not continuous

K continuous $\Rightarrow$ $\int K(x,y) f(\xi)\,d\xi$ compact

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$ yes: $r^{-n+\varepsilon}$
- There exist $C > 0$, $\alpha \in (0, n]$ such that no: r^{-n}

$$|K(x,y)| \le C|x-y|^{\alpha-n} \qquad (x \ne y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 3rd ed. Thm. 2.29])

K weakly singular. Then

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

is compact on $C(G)$, where $\mathrm{cl}(G^\circ) = G$.

Weakly singular: Proof Outline

$\int_0^1 \frac{1}{x}\,dx$

$\int_0^1 \frac{1}{x^{0.99}}$

Outline the proof of 'Weakly singular kernel $\Rightarrow$ compact'.

$$A\varphi(x) = \int K(x,y)\varphi(y)\,dy$$

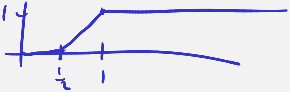
$$\int_{B(0,d)} |x-y|_2^{\alpha-n}\,dy = \omega_n \int_0^d \rho^{\alpha-n}\rho^{n-1}\,d\rho = \omega_n \int_0^d \rho^{\alpha-1}\,d\rho = \frac{d^\alpha}{\alpha}\omega_n$$

$\alpha > 0$

ω_n: surface area of the sphere in $\mathbb{R}^n$

bounded ✓

$h(x) :=$



$h \in C^0$

$$K_n(x,y) := h(n|x-y|)\,K(x,y) \in C^0 \qquad K_n \to K$$

$$\hookrightarrow (A_n\varphi)(x) := \int K_n(x,y)\varphi(y)\,dy \qquad A_n \to A \text{ in norm}$$

A_n compact $\Rightarrow A$ compact.

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, $\partial\Omega$ is C^1 (what does that mean?)

Definition (Weakly singular kernel (on a surface))

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \le C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \ne y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 3rd ed. Thm. 2.30])

K weakly singular on $\partial\Omega$. Then $(A\phi)(x) := \int_{\partial\Omega} K(x,y)\phi(y)dy$ is compact on $C(\partial\Omega)$.

Q: Has this estimate gotten worse or better?

Outline

Riesz Theory (I)

Still trying to solve

$$L\phi := \underbrace{(I - A)}_{L}\phi = \phi - A\phi = f$$

with A compact.

Theorem (First Riesz Theorem [Kress LIE 3rd ed., Thm. 3.1]**)**

$N(L)$ is finite-dimensional.

Questions:

- What is $N(L)$ again? nullspace$(L) = \{\varphi : L\varphi = 0\}$
- Why is this good news?

Riesz First Theorem: Proof Outline

Show it.

$\varphi \in N(L) \Leftrightarrow L\varphi = 0 \quad \Leftrightarrow \quad (I - A)\varphi = 0 \quad \Leftrightarrow \quad \varphi - A\varphi = 0$

$\Leftrightarrow \quad \varphi = A\varphi$

$A|_{N(L)} = I$

$\Rightarrow N(L)$ must be $\dim(N(L)) < \infty$.

Riesz Theory (II)

Theorem (Riesz theory [Kress LIE 3rd ed., Thm. 3.4])

A compact. Then:

- $(I - A)$ *injective* $\Leftrightarrow$ $(I - A)$ *surjective*
  - *It's either bijective or neither s nor i.*
- *If* $(I - A)$ *is bijective,* $(I - A)^{-1}$ *is bounded.*

one-to-one

onto

" $\Rightarrow$ "

Rephrase for solvability:

uniqueness $\Rightarrow$ existence (civilized)

Key shortcoming?

nullspace $\Rightarrow$ we're toast.

Riesz Theory: Boundedness Proof Outline

Assuming $(I - A)$ is bijective, show that $(I - A)^{-1}$ is bounded.

Assume L^{-1} unbounded.

Can pick f_n with $\|f_n\| = 1$ with $\|L^{-1} f_n\| > n$.

$g_n = f_n / \|L^{-1} f_n\| \to 0$ $\qquad \varphi_n = L^{-1} f_n / \|L^{-1} f_n\|$

$0 \leftarrow g_n = \frac{f_n}{\|L^{-1} f_n\|} = L\varphi_n = \varphi_n - A\varphi_n$ $\qquad \|\varphi_n\| = 1$

Pick subseq of φ_n: $\varphi_{n(k)}$: $A\varphi_{n(k)} \to \varphi$

(Quite a bit more to the proof: Riesz' second, third theorems, Riesz numbers, . . .)

$A\varphi = \varphi \Rightarrow \varphi \in N(L)$

contradicts injectivity.

Hilbert spaces

Hilbert space: Banach space with a norm coming from an *inner product*:

$$(\alpha x + \beta y, z) = ? \qquad \alpha(x, z) + \beta(y, z)$$

$$(x, \alpha y + \beta z) = ?$$

$$(x, x)? \; = 0 \iff x = 0 \qquad \|x\| = \sqrt{(x,x)}$$

$$(y, x) = ? \quad (x, y)$$

$$\cancel{\|x\| = (x,x)} \qquad \cancel{\|2x\| = 4(x,x)}$$

G compact

Is $C^0(G)$ a Hilbert space?

$$(f,g) = \int f(x) g(x)\, dx \qquad (f,f) = 0$$

$$= \int f^2\, dx \iff f = 0$$

Name a Hilbert space of functions.

L^2 $\qquad$ C^0 not closed with L^2 norm.

Continuous and Square-Integrable

Can we carry over $C^0(G)$ boundedness/compactness results to $L^2(G)$?

X, Y normed spaces with a scalar product so that $|(\phi, \psi)| \leq \|\phi\| \, \|\psi\|$ for $\phi, \psi \in X$.

Theorem (Lax dual system [Kress LIE 3rd ed. Thm. 4.13])

Let $U \subseteq X$ be a subspace and let $A : X \to Y$ and $B : Y \to X$ be bounded linear operators with

$$(A\phi, \psi) = (\phi, B\psi) \qquad (\phi \in U, \psi \in Y).$$

Then $A : U \to Y$ is bounded with respect to $\|\cdot\|_s$ induced by the scalar product and $\|A\|_s^2 \leq \|A\| \, \|B\|$.

Based on this, it is also possible to carry over compactness results.

Adjoint Operators

Definition (Adjoint oeprator)

A^* called adjoint to A if

$$(Ax, y) = (x, A^* y)$$

for all x, y.

Facts:

- A^* unique
- A^* exists
- A^* linear
- A bounded $\Rightarrow$ A^* bounded
- A compact $\Rightarrow$ A^* compact

Adjoint Operator: Observations?

What is the adjoint operator in finite dimensions? (in matrix representation)

What do you expect to happen with integral operators?

Adjoint of the single-layer?

Adjoint of the double-layer?

Fredholm Alternative

Theorem (Fredholm Alternative [Kress LIE 3rd ed. Thm. 4.17])

$A : X \to X$ compact. Then either:

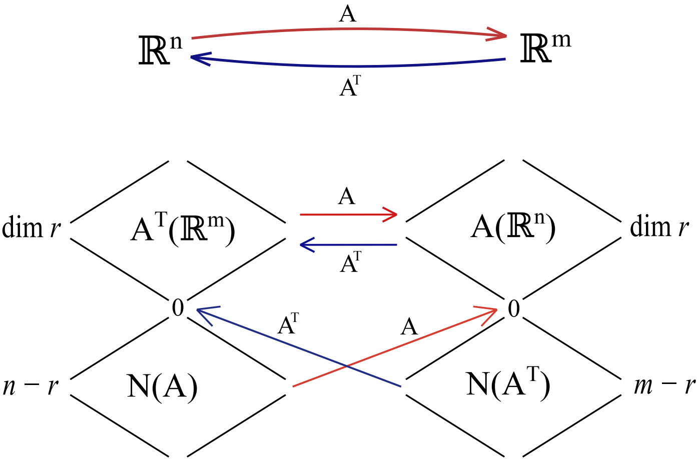
- *$I - A$ and $I - A^*$ are bijective*

  or:

- $\dim N(I - A) = \dim N(I - A^*)$
- $(I - A)(X) = N(I - A^*)^\perp$
- $(I - A^*)(X) = N(I - A)^\perp$

Seen these statements before?

Fundamental Theorem of Linear Algebra



[Credit: Wikipedia]