Ann:
- Project proposals
  - Pres.: May 7, 9
  - Due: May 10
- HW4

Goals:
- Spectral review
- Harmonic functions $\Delta u = 0$
  - One-line proofs

Review:
- (I - A) $\phi = 0$ solvability
  - Riesz
  - Fredholm
  
  \[(I - A)(x) = N(I - A^*)^\perp\]
  
  \[(I - A)^* = I^* - A^* = I - A^*\]
  
  \[(I_x, y) = (x, I_y)\]
  
- Spectral
Spectral Theory: Terminology

A : X → X bounded, λ is a ... value:

Definition (Eigenvalue)
There exists an element \( \phi \in X, \phi \neq 0 \) with \( A\phi = \lambda \phi \).

Definition (Regular value)
The “resolvent” \( (\lambda I - A)^{-1} \) exists and is bounded.

Can a value be regular and “eigen” at the same time?

\[ \forall \]

What’s special about \( \infty \)-dim here?

\[ \lambda \in \mathbb{D} \text{ is neither eigen nor regular} \]
Resolvent Set and Spectrum

**Definition (Resolvent set)**

\[ \rho(A) := \{ \lambda \text{ is regular} \} \]

\[ \subseteq \mathbb{C} \iff \exists \lambda : (\lambda I - A)^{-1} \text{ exists} \]

**Definition (Spectrum)**

\[ \sigma(A) := \mathbb{C} \setminus \rho(A) \]

\[ \subseteq \mathbb{C} \iff (\lambda : (\lambda I - A)^{-1} \text{ is not defined or unbounded}) \]
Theorem

\( A : X \to X \) compact linear operator, \( X \) \(\infty\)-dim.

Then:

\[ 0 \in \sigma(A) \]

\[ \sigma(A) \setminus \{0\} \text{ consists only of eigenvalues} \]

\[ \sigma(A) \setminus \{0\} \text{ is at most countable} \]

\[ \sigma(A) \text{ has no accumulation point except for } 0 \]
Show the first part.

$0 \in \sigma(A)$

$\Leftarrow$ $0$ is not a regular value.

$\Rightarrow$ $A^{-1}$ does not exist or is not bounded.

Suppose $A^{-1}$ existed and is bounded, $I = AA^{-1}$ compact $\Rightarrow$

Show second part.

Let $\lambda \in \sigma(A)$ $\Leftarrow$ $(\lambda I - A)^{-1}$ does not exist and is not bounded.

$\lambda I - A$ is not bijective.

$\exists \Phi : (\lambda \Phi - A) \cdot \Theta$ is not injective.

$\Rightarrow$ $\Phi$ is eigen.

Theorem: $\Phi$ has nullspace
If $\forall x, y : Ax = Ay \Rightarrow x = y$,

$\Rightarrow \forall x, y : Ax + Ay \Rightarrow x + y = 0$

$A(x - y) = 0 \Rightarrow x - y = 0$

$\forall z : A\, z = 0 \Rightarrow z = 0$

$\Rightarrow \quad N(A) = \{0\}$
Spectral Theory of Compact Operators: Implications

Rephrase last two: how many eigenvalues with $| \cdot | \geq R$?

Recap: What do compact operators do to high-frequency data?

Don’t confuse $I - A$ with $A$ itself!
Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

**Singular Integrals and Potential Theory**
- Singular Integrals
- Green's Formula and Its Consequences
- Jump Relations

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs
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Recap: Layer potentials

\[(S\sigma)(x) := \int_{\Gamma} G(x - y)\sigma(y) ds_y\]

\[(S'\sigma)(x) := PV \hat{n} \cdot \nabla_x \int_{\Gamma} G(x - y)\sigma(y) ds_y\]

\[(D\sigma)(x) := PV \int_{\Gamma} \hat{n} \cdot \nabla_y G(x - y)\sigma(y) ds_y\]

\[(D'\sigma)(x) := f.p. \hat{n} \cdot \nabla_x \int_{\Gamma} \hat{n} \cdot \nabla_y G(x - y)\sigma(y) ds_y\]

**Definition (Harmonic function)**

\(\Delta u = 0\)

Where are layer potentials harmonic?

\[G = \frac{1}{2\pi i} \log \frac{1}{z} \quad \text{away from } \Gamma\]
\[ \Delta \int_{\mathcal{P}} \mathcal{G}(x,y) \sigma(y) \, d\mathbf{s}_y \]

\[ = \int \frac{\delta(x-y)}{\delta(x-y)} \sigma(y) \, d\mathbf{s}_y \]

\[ = \int \delta(x-y) \sigma(y) \, d\mathbf{s}_y = 0 \]
On the double layer again

Is the double layer *actually* weakly singular? Recap:

**Definition (Weakly singular kernel)**

- $K$ defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n - 1]$ such that

\[
|K(x, y)| \leq C|x - y|^\alpha - n + 1 \quad (x, y \in \partial \Omega, \ x \neq y)
\]

weakly singular $\Rightarrow$ A compact $\frac{1}{\sqrt{0.999}}$ is OK

$\frac{1}{\sqrt{2}}$ is not
Actual Singularity in the Double Layer (2D)

$$\frac{\partial}{\partial x} \log(|0 - x|) = \frac{x}{x^2 + y^2}$$

- Singularity with approach on $y = 0$?
- Singularity with approach on $x = 0$?

2D only: the double layer has a removable singularity
Cauchy Principal Value

But I don’t want to integrate across a singularity! → punch it out.

Problem: Make sure that what’s left over is well-defined

\[ \int_{-1}^{1} \frac{1}{x} \, dx? \]

\[ \text{PV} \int_{-1}^{1} \frac{1}{x} \, dx = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{1}{x} \, dx + \int_{\varepsilon}^{1} \frac{1}{x} \, dx \]

defined via cancellation of blow-ups.

\[ \text{PV} \int_{-1}^{1} \frac{1}{x} \, dx = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{1}{x} \, dx + \int_{\varepsilon}^{1} \frac{1}{x} \, dx \]

\[ \text{PV} \int_{-1}^{1} \frac{1}{x} \, dx \neq \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{1}{x} \, dx + \int_{\varepsilon}^{1} \frac{1}{x} \, dx \]
Principal Value in \( n \) dimensions

Integration Contour

Again: Symmetry matters!

has to be a circle.

What about even worse singularities?

f.p.
Recap: Layer potentials

\[(S\sigma)(x) := \int_{\Gamma} G(x - y)\sigma(y)dy\]

\[(S'\sigma)(x) := \text{PV} \hat{n} \cdot \nabla_x \int_{\Gamma} G(x - y)\sigma(y)dy\]

\[(D\sigma)(x) := \text{PV} \int_{\Gamma} \hat{n} \cdot \nabla_y G(x - y)\sigma(y)dy\]

\[(D'\sigma)(x) := \text{f.p.} \hat{n} \cdot \nabla_x \int_{\Gamma} \hat{n} \cdot \nabla_y G(x - y)\sigma(y)dy\]

Important for us: Recover ‘average’ of interior and exterior limit without having to refer to off-surface values.

(to be shown)
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Green’s Theorem

Ω bounded

Theorem (Green’s Theorem [Kress LIE 2nd ed. Thm 6.3])

\[
\int_{\Omega} u \Delta v + \nabla u \cdot \nabla v = \int_{\partial \Omega} u (\hat{n} \cdot \nabla v) \, ds
\]

\[
\int_{\Omega} u \Delta v - v \Delta u = \int_{\partial \Omega} u (\hat{n} \cdot \nabla v) - v (\hat{n} \cdot \nabla u) \, ds
\]

If \( \Delta v = 0 \) and \( u = 1 \), then

\[
\int_{\partial \Omega} \hat{n} \cdot \nabla v = ?
\]

\[
\int_{\Omega} u \Delta v - v \Delta u = \int_{\partial \Omega} u (\hat{n} \cdot \nabla v) - v (\hat{n} \cdot \nabla u) \, ds
\]

\( \neq 0 \)
Green’s Formula

What if $\triangle v = 0$ and $u = G(|y - x|)$ in Green’s second identity?

$$\int_{\Omega} u \triangle v - v \triangle u = \int_{\partial \Omega} u(\hat{n} \cdot \nabla v) - v(\hat{n} \cdot \nabla u) ds$$

Can you write that more briefly?

$$-\int_{\Omega} v(y) \delta(x-y) dy = \int_{\partial \Omega} G(x-y) \partial_n v - \int_{\partial \Omega} v$$

$$\int_{\Omega} v(x) \chi \in \mathcal{R}$$

$$0 \times \mathbb{R} = S(\partial_n u) - D(v)$$
Green’s Formula (Full Version)

Ω bounded

Theorem (Green’s Formula [Kress LIE 2nd ed. Thm 6.5])

If $\triangle u = 0$, then

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} 
  u(x) & x \in \Omega, \\
  \frac{u(x)}{2} & x \in \partial\Omega, \\
  0 & x \notin \Omega.
\end{cases}$$
Suppose I know ‘Cauchy data’ \((u|_{\partial \Omega}, \hat{n} \cdot \nabla u|_{\partial \Omega})\) of \(u\). What can I do?

What if \(\Omega\) is an exterior domain?

What if \(u = 1\)? Do you see any practical uses of this?