April 23, 2024
Announcements

- HW4 due
- Projects

Goals

- $N(\frac{1}{2} + 0) \leq 3\text{pm} \{1\}$
- ext. Dirichlet
- Helmholtz

Review

- UVP uniqueness
- IE 'uniqueness'
Uniqueness of Integral Equation Solutions

Theorem (Nullspaces [Kress LIE 3rd ed. Thm 6.21])

\[
\begin{align*}
\text{Fredholm alt.:} & \quad (I - A)(x) = (N(-A^*)^*) \\ 
\text{int. Neuman} \Rightarrow (I/\ell + S')(x) \perp N(\frac{I}{\ell} + D) = \text{span}\{1\} \\ 
\text{exl. Dirichlet} \Rightarrow (I/\ell + D)(x) \perp N(\frac{1}{\ell} + S') = \text{span}\{\psi\}, \quad \text{where}\int \psi \neq 0.
\end{align*}
\]
Show $N(I/2 - D) = \{0\}$. 

(∀ $\psi$: $\nabla \cdot \psi = 0$, $u(I) = D\psi$. 
\lim_{r \to 0} u = 0$ on int. (vol.) 
$(\partial_n u) = 0$ 

$u = 0$ on ext. (vol.) 

$\Rightarrow \psi = 0.$
IE Uniqueness: Proofs (2/3)

Show $N(I/2 - S') = \{0\}$. 
IE Uniqueness: Proofs (3/3)

Show \( N(I/2 + D) = \text{span}\{1\} \).

\[ \phi \text{ s.t. } \frac{\partial^2 \phi}{\partial x^2} + D \phi = 0. \quad u(x) = 1, \phi(x) \Rightarrow u^+ = 0. \]

\[ \lim_{x \to \pm \infty} u = 0 \quad \text{in ext. (W1)} \]

\[ (\Theta^+_u) = 0 \Rightarrow (\Theta^- u) = 0 \Rightarrow u \equiv \text{const. in int. vol.} \]

\[ \varphi = [D\varphi] = u^+ - u^- = 0 - \text{const.} \Rightarrow \varphi \in \text{span } \{1\} \]

\[ \square \checkmark \text{ because of } D1 \]

What conditions on the RHS do we get for int. Neumann and ext. Dirichlet?

\[ \ldots \text{ (see above) } \]
Patching up Exterior Dirichlet

Problem: \( N(1/2 + S') = \{ \psi \} \ldots \) do not know \( \psi \). Assume \( 0 \in \Omega \).

\[
K(x, y) = \nabla_y G(x, y) \sim \tilde{K}(x, y) = \nabla_y \tilde{G}(x, y) + \frac{1}{|x|^{n-2}}
\]

\[
u(x) = \int_{\Gamma} \tilde{K}(x, y) \sigma(y) \, ds_y = D \circ + \frac{1}{|x|^{n-2}} \int_{\Sigma} \sigma(y) \, ds_y
\]

- Solves PDE, stays compact, no admissible jumps, ext. limit \( \nu(x) \)
- Ext. Dirichlet uniqueness \( \Rightarrow u \equiv 0 \) in ext. vol.

\[
0 = |x|^{1-n/2} u(x) \Rightarrow \sigma + O(\frac{1}{r}) \to \sigma^\circ \quad \text{as} \ x \to \infty
\]

if \( D \circ = O(\frac{1}{r}) \) in 3D

\[
\frac{\sigma^\circ}{\sigma^\circ} = 0 \Rightarrow \sigma \in N(\frac{1}{2} + D) - \text{span } \{1\} \Rightarrow \sigma^\circ = 0
\]

\( \Rightarrow \sigma = 0 \).
What’s the problem?

$D_1 = \{ -\frac{1}{2} \}$
Domains with Corners (II)

At corner $x_0$: (2D)

$$
\lim_{x \to x_0 \pm} = \int_{\partial \Omega} \hat{n} \cdot \nabla_y G(x, y)\phi(y) ds_y \pm \frac{1}{2} \left< \text{opening angle on } \pm \text{ side} \right> \phi
$$

Name some problems.

$$
\lim_{x \to \Omega^\pm} u(x) = D\sigma \begin{cases} 
-\sigma/2 \\
-\sigma \cdot ?
\end{cases} \rightarrow \text{not second kind?}
$$

Workarounds?

- $I^+$ bounded $\&$ compact
- $C$

Numerically: Needs consideration, can drive up cost through refinement.
Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

- Laplace
- Helmholtz
- Calderon identities

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs
Where does Helmholtz come from?

Derive the Helmholtz equation from the wave equation \( \partial_t^2 U = c^2 \Delta U \).

Q: What is \( c \)?

\[ c = \text{prop speed} \quad [c] = \frac{m}{s} \]

\( U(x,t) = u(x) e^{-i\omega t} \)

\[ u(x) (-i\omega)^2 e^{-i\omega t} = c^2 e^{-i\omega t} \Delta u(x) \]

\[ -\omega^2 u(x) = c^2 \Delta u(x) \]

\[ \Delta \omega^2 \left( \frac{\omega^2}{c^2} \right) u(x) = 0 \]

\[ \frac{1}{k} \]

*wave number*
Helmholtz vs. Yukawa

Helmholtz Equation
- $\Delta u + k^2 u(x) = 0$
- Indefinite operator
- Oscillatory solution
- Difficult to solve, especially for large $k$

Yukawa Equation
- $-\Delta u + k^2 u(x) = 0$
- Positive definite operator
- Smooth solutions
- ‘Screened Coulomb’ interaction
- Generally quite simple to solve

$-\Delta$ is pos. def., $\int \Delta u \cdot \nabla \varphi \geq 0$
The prototypical Helmholtz BVP: A Scattering Problem

Ansatz:

\[ u^{\text{tot}} = u + u^{\text{inc}} \]

Solve for scattered field \( u \).
Helmholtz: Some Physics

Physical quantities:

- Velocity potential: \( U(x, t) = u(x)e^{-i\omega t} \)
  (fix phase by e.g. taking real part)
- Velocity: \( v = (1/\rho_0)\nabla U \)
- Pressure: \( p = -\partial_t U = i\omega u e^{-i\omega t} \)
  - Equation of state: \( p = f(\rho) \)

What’s \( \rho_0 \)?

"base density is lin. of Euler"

What happens to a pressure BC as \( \omega \to 0 \)?

"disappears"
Helmholtz: Boundary Conditions

Interfaces between media: What’s continuous?

- **Sound-soft**: Scatterer “gives”
  - Pressure remains constant in time
  - \( u = f \rightarrow \text{Dirichlet} \)

- **Sound-hard**: Scatterer “does not give”
  - Pressure varies, same on both sides of interface
  - \( \hat{n} \cdot \nabla u = 0 \rightarrow \text{Neumann} \)

- **Impedance**: Some pressure translates into motion
  - Scatterer “resists”
  - \( \hat{n} \cdot \nabla u + ik \lambda u = 0 \rightarrow \text{Robin} \ (\lambda > 0) \)

- **Sommerfeld** radiation condition: allow only outgoing waves (\( n \)-dim)
  \[
r^{\frac{n-1}{2}} \left( \frac{\partial}{\partial r} - ik \right) u(x) \rightarrow 0 \quad (r \rightarrow \infty)
  \]

Many interesting BCs \( \rightarrow \) many IEs! :)