Announcements:

HW 2 → due in 2 weeks

HW1 due tomorrow

Goals:

- compare num. rank w. error estimates from Taylor
- alternatives to Taylor
- "expansions" using LA

Review:

Local:

$$\text{Err} \sim \left(\frac{d(c, t.\text{target})}{d(c, c.\text{source})} \right)^{k+1}$$

Mpole:

$$\text{Err} \sim \left(\frac{d(c, t.\text{source})}{d(c, c.\text{target})} \right)^{k+1}$$

$$\sum_{|\vec p| \le k} \frac{\partial^{\vec p} f}{\vec p!} h^{\vec p}$$

$$\#\ \text{terms} = O(k^d)$$

2D:

On Rank Estimates

So how many terms do we need for a given precision ε?

$$\varepsilon \sim \left(\frac{d(c, f.\text{target})}{d(c, c.\text{source})}\right)^{k+1} = \rho^{k+1}$$

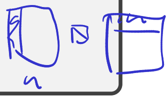
\# terms $K \approx k^2 \rightsquigarrow k = \sqrt{K}$ $\qquad \varepsilon \approx \rho^{\sqrt{K}+1}$

$$\log \varepsilon \approx (\sqrt{K}+1) \log \rho$$

$$\sqrt{K}+1 = \frac{\log \varepsilon}{\log \rho}$$

$$K = \left(\frac{\log \varepsilon}{\log \rho} - 1\right)^2$$

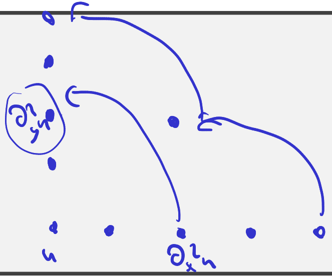
$$A = U \Sigma V^T$$

$\approx$ 

Demo: Checking rank estimates

Estimated vs Actual Rank

Our rank estimate was off by a power of $\log \varepsilon$. What gives?

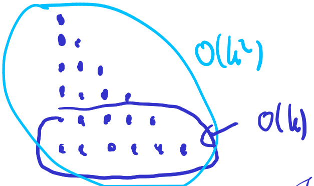


$u = u_0|_\Gamma$

$\Delta u = 0$

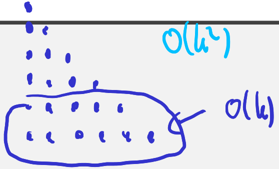
$(\partial_x^2 + \partial_y^2) u = 0$

$\Leftrightarrow \partial_x^2 u = -\partial_y^2 u$



Taylor and PDEs

Look at $\partial_x^2 G$ and $\partial_y^2 G$ in the multipole demo again. Notice anything?

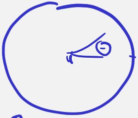


Being Clever about Expansions

$\Delta u + k^2 u = 0$

How could one be clever about expansions? (i.e. give examples)

Way 1

- ▷  $u = u_0|_{\partial B(0,r)}$ $\quad u_0(\theta) = \sum_{j=-n}^{n} \alpha_n e^{ij\theta}$

  (rep. function on surface (e.g. using spherical harmonics)
- ▷ do polar coord sep. of variables on the PDE
- ▷ find formula to evaluate e at any point (r, θ)

Way 2

- ▷ If f is complex differentiable, then $\Delta \operatorname{Re} f = 0$
- ▷ use complex-var Taylor

Expansions for Helmholtz

How do expansions for other PDEs arise?

DLMF 10.23.6 shows 'Graf's addition theorem':

$$H_0^{(1)}\left(\kappa\left\|x-y\right\|_2\right)=$$
$$\sum_{\ell=-\infty}^{\infty}\underbrace{H_\ell^{(1)}\left(\kappa\left\|y-c\right\|_2\right)e^{i\ell\theta'}}_{\text{singular}}\underbrace{J_\ell\left(\kappa\left\|x-c\right\|_2\right)e^{-i\ell\theta}}_{\text{nonsingular}}$$

where $\theta=\angle(x-c)$ and $\theta'=\angle(x'-c)$.

Can apply same family of tricks as with Taylor to derive multipole/local expansions.

Outline

Making Multipole/Local Expansions using Linear Algebra

Actual expansions cheaper than LA approaches. Can this be fixed?

Compare costs for this situation:

S sources

T targets

interaction rank $K \ll \min(S, T)$

Expansions:

Form: (compute coeffs) $O(KS)$

Evaluate $O(KT)$

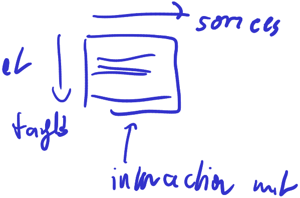
$\}\ O(K(S+T))$

Cost for LA:

Form matrix $O(ST)$

The Proxy Trick

For locals,
find row subset



Idea: *Skeletonization using Proxies*

Demo: Skeletonization using Proxies

Q: What error do we expect from the proxy-based multipole/local 'expansions'?

Why Does the Proxy Trick Work?

In particular, how general is this? Does this work for any kernel?