

Announcements:

HW 2 \rightarrow due in 2 weeks

HW1 due tomorrow

Goals:

- compare num. rank w. error estimates from Taylor
- alternatives to Taylor
- "expansions" using LA

Review:

Local r

$$\bar{\text{Err}} \sim \left(\frac{d(c, F.\text{target})}{d(c, c.\text{source})} \right)^{k+1}$$

Npoles

$$\bar{\text{Err}} \sim \left(\frac{d(c, F.\text{source})}{d(r, c.\text{target})} \right)^{k+1}$$

$$\sum_{|p| \leq k} \frac{\vec{D}^p F}{h^{|p|}}$$

$$\# \text{ terms} = \mathcal{O}(k^d)$$

$$2D: \dots \dots \dots (4D)$$

On Rank Estimates

So how many terms do we need for a given precision ϵ ?

$$\epsilon \sim \left(\frac{d(c, f, \text{target})}{d(c, c, \text{source})} \right)^{k+1} = \rho^{k+1}$$

$$\# \text{ terms } k \approx k^2 \quad \rightsquigarrow \quad k \approx \sqrt{k} \quad \epsilon \approx \rho^{\sqrt{k+1}}$$

$$\log \epsilon \approx (\sqrt{k+1}) \log \rho$$

$$\sqrt{k+1} = \frac{\log \epsilon}{\log \rho}$$

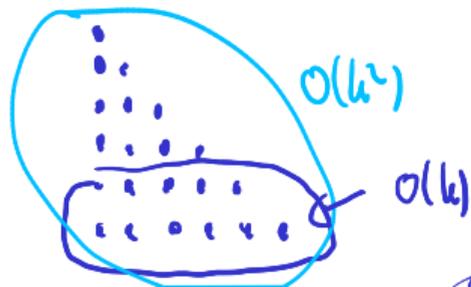
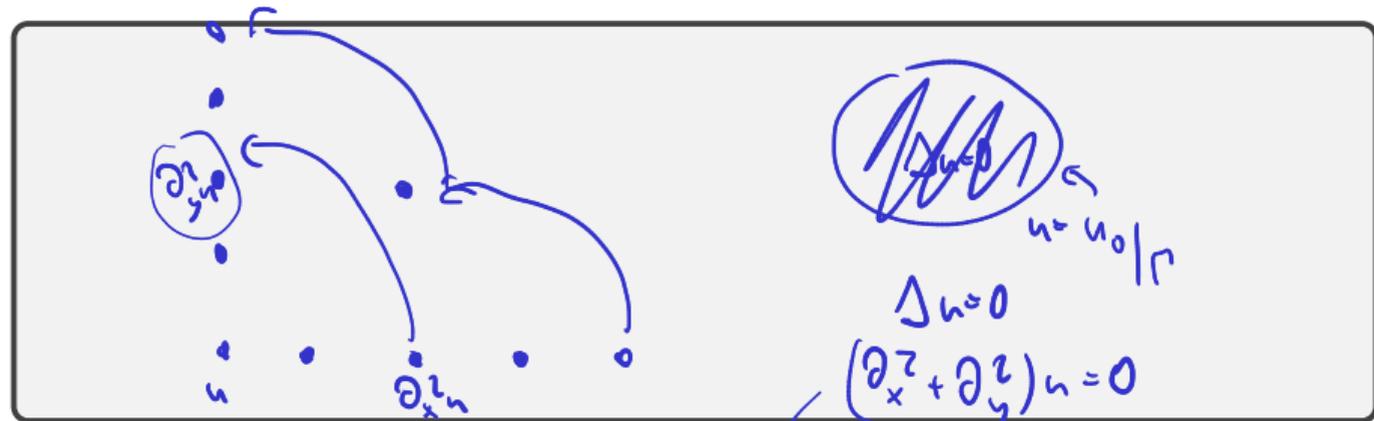
$$k = \left(\frac{\log \epsilon}{\log \rho} - 1 \right)^2$$

$$A = U \Sigma V^T$$

\approx 

Estimated vs Actual Rank¹

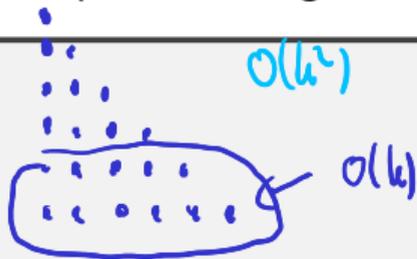
Our rank estimate was off by a power of $\log \varepsilon$. What gives?



$$\Delta h = 0$$
$$n = n_0 / \Gamma$$
$$\Delta h = 0$$
$$(\partial_x^2 + \partial_y^2) h = 0$$
$$\Rightarrow \partial_x^2 h = -\partial_y^2 h$$

Taylor and PDEs

Look at $\partial_x^2 G$ and $\partial_y^2 G$ in the multipole demo again. Notice anything?

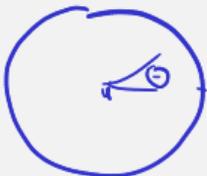


Being Clever about Expansions

$$\Delta u + \kappa^2 u = 0$$

How could one be clever about expansions? (i.e. give examples)

Way 1

\triangleright  $u = u_0 |_{DB(0, r)}$ $u_0(\theta) = \sum_{j=-n}^n \alpha_n e^{ij\theta}$

(rep. function on surface (e.g. using spherical harmonics))

\triangleright do polar coord sep. of variables on the PDE

\triangleright find formula to evaluate u at any point (r, θ)

Way 2

\triangleright If f is complex differentiable, then $\Delta \operatorname{Re} f = 0$

\triangleright use complex-valued Taylor

Expansions for Helmholtz

How do expansions for other PDEs arise?



DLMF 10.23.6 shows 'Graf's addition theorem':

$$\sum_{\ell=-\infty}^{\infty} \underbrace{H_{\ell}^{(1)}(\kappa \|y - c\|_2) e^{i\ell\theta'}}_{\text{singular}} \underbrace{J_{\ell}(\kappa \|x - c\|_2) e^{-i\ell\theta}}_{\text{nonsingular}}$$

$H_0^{(1)}(\kappa \|x - y\|_2) =$

where $\theta = \angle(x - c)$ and $\theta' = \angle(x' - c)$.

Can apply same family of tricks as with Taylor to derive multipole/local expansions.

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Local Expansions

Multipole Expansions

Rank Estimates

Proxy Expansions

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Making Multipole/Local Expansions using Linear Algebra

Actual expansions cheaper than LA approaches. Can this be fixed?

Compare costs for this situation:

S sources

T targets

interaction rank $K \ll \min(S, T)$

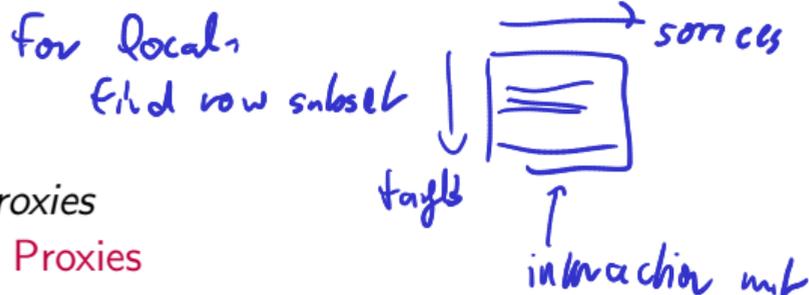
Expansions:

Form: (compute coeffs) $O(KS)$
Evaluate $O(KT)$ } $O(K(S+T))$

Cost for LA:

Form matrix $O(ST)$

The Proxy Trick



Idea: Skeletonization using Proxies

Demo: Skeletonization using Proxies

Q: What error do we expect from the proxy-based multipole/local 'expansions'?



Why Does the Proxy Trick Work?

In particular, how general is this? Does this work for any kernel?

