Ann:

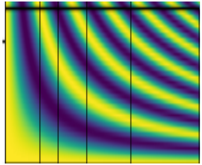
- HW 3 due tomorrow
- Project proposals

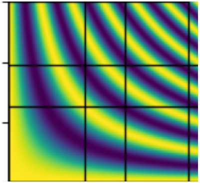
Goals:

- Butterfly
- PDE solvers using our machinery
- Theory

Review:

- Butterfly





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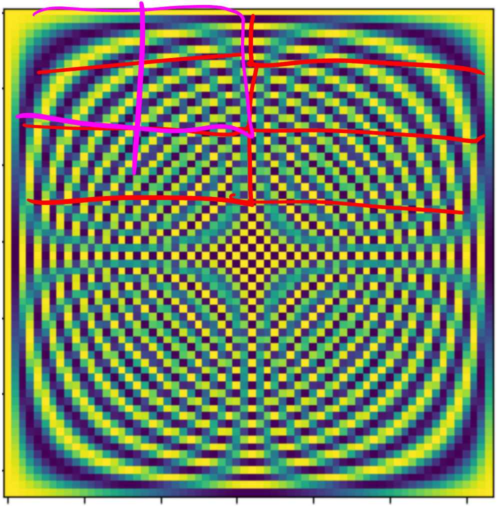
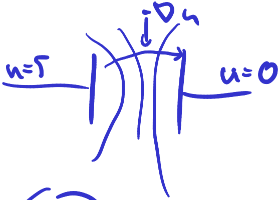


# PDEs: Simple Ones First, More Complicated Ones Later




**Laplace**

$\triangle u = 0$

*Applications:*

- Steady-state $\partial_t u = 0$ of wave propagation, heat conduction
- Electric potential $u$ for applied voltage
- Minimal surfaces/“soap films”
- $\nabla u$ as velocity of incompressible/potential flow

Yakawa

**Helmholtz**

$\triangle u + k^2 u = 0$

Assume time-harmonic behavior $\tilde{u}_{(x,t)} = e^{\pm i\omega t}u(x)$ in time-domain wave equation:

$$\partial_t^2 \tilde{u} = \triangle \tilde{u}$$

$\hat{u}(x,t)$

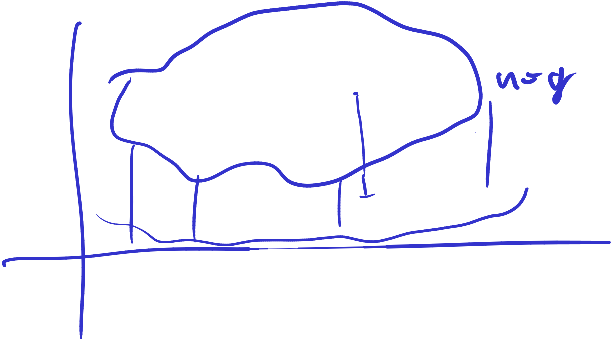
*Applications:*

- Propagation of sound
- Electromagnetic waves

$$\partial_t^2(\underbrace{e^{-i\omega t}u(x)}_{\tilde u}) = (-i\omega)^2\tilde{u}$$

$-\omega^2\tilde u = \Delta\tilde u$

$\Delta\hat u + \omega^2\hat u = 0$

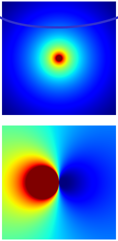


# Fundamental Solutions

**Laplace**

$-\triangle u = \delta$

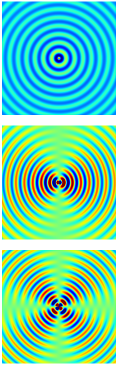
Def. of Free Space Green's function



$\partial_x$

**Helmholtz**

$\triangle u + k^2 u = \delta$



$\partial_x$

$\partial_y$

aka. *Free space Green's Functions*

How do you assign a precise meaning to the statement with the $\delta$-function?

$\boxed{\text{weakly: multiply by test func } \in C^\infty, \text{ integrate}}$

# Green's Functions

$\Delta u = 0 \leftarrow$ Laplace

$\Delta u = f \leftarrow$ Poisson

Why care about Green's functions?

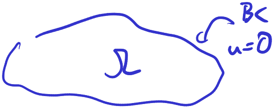
$$u(x) = (G * f)(x) = \int G(x-y)\, f(y)\, dy$$

$$\Delta u(x) = \Delta \int G(x-y)\, f(y)\, dy = \int \underbrace{\Delta G(x-y)}_{\delta(x-y)} f(y)\, dy = f(x)$$

What is a non-free-space Green's function? I.e. one for a specific domain?

$u(x) = \int G_{\text{domain}}(x,y)\, f(y)\, dy$ solves and $\Delta u = f$, $u = 0$ on $\partial\Omega$

"domain Green's function"



## Green's Functions (II)

Why not just use domain Green's functions?

domain, BC - specific hard to find

What if we don't know a Green's function for our PDE... at all?

· could use Green's func for only the highest-order bit of the PDE.

# Fundamental Solutions

**Laplace**

$$G(x) = \begin{cases} \frac{1}{-2\pi} \log |x| & \text{2D} \\ \frac{1}{4\pi} \frac{1}{|x|} & \text{3D} \end{cases}$$

$$\frac{\partial}{\partial_x} G(x)$$

**Helmholtz**

Hankel f. of 1st kind of order 0

$$G(x) = \begin{cases} \frac{i}{4} H_0^1(k|x|) & \text{2D} \\ \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} & \text{3D} \end{cases}$$

$$\frac{\partial}{\partial_x} G(x)$$

# Layer Potentials (I)

$\Gamma$ is a surface $\subset \mathbb{R}^d$ $\Delta u = 0$ on $\Omega$

$u = g$ on $\partial\Omega$ $\Omega$

Let $G_k$ be the Helmholtz kernel ($k = 0 \rightarrow$ Laplace).

$$(S_k\sigma)(x) = \int_\Gamma G_k(x-y)\,\sigma(y)\,dS_y \quad \leftarrow \text{"single layer"}$$

$$(x \in \Gamma) \quad (S'_k\sigma)(x) = \partial_{n_x} \int_\Gamma G_k(x-y)\,\sigma(y)\,dS_y$$

$$(D_k\sigma)(x) = \int_\Gamma \partial_{n_y} G_k(x-y)\,\sigma(y)\,dS_y$$

$$(D'_k\sigma)(x) = \partial_{n_x} \int_\Gamma \partial_{n_y} G_k(x-y)\,\sigma(y)\,dS_y$$

These operators map function $\sigma$ on $\Gamma$ to...

Functions in all of space (including $\Gamma$!)

# Layer Potentials (II)

Called *layer potentials*:

- $S$ is called the *single-layer potential*
- $D$ is called the *double-layer potential*
- $S''$ (and higher) analogously

(Show pictures using `pytential/examples/layerpot.py`, observe continuity properties.)

Alternate ("standard") nomenclature:

| Ours | Theirs |
|---|---|
| $S$ | $V$ |
| $D$ | $K$ |
| $S'$ | $K'$ |
| $D'$ | $T$ |

# How does this actually solve a PDE?

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u|_\Gamma = f|_\Gamma.$$

$u(x) = S_\Gamma \sigma(x)$ wrt an unknown $\sigma$.

Take int. limit: $u(x)|_{\Gamma^-} = f(x)$

$$\underbrace{\int_\Gamma G_0(x-y)\,\sigma(y)\,dS_y}_{} = f(x) \quad (x \in \Gamma)$$

pot

discretize $\hookrightarrow$ $S\sigma = f$

# IE BVP Solve: Observations (I)

Observations:

- One can choose representations relatively freely. Only constraints:
  - Can I get to the solution with this representation?
    I.e. is the solution I'm looking for represented?
  - Is the resulting integral equation solvable?

Q: How would we know?