Aim

Project proposals

Goals

Review

Layer potentials

↳ PDE solvers

↳ Integral equations "Fredholm" "of the second kind"

↳ "compactness"

Layer Potentials (I)

$G_k(x-y)$

Let G_k be the Helmholtz kernel ($k = 0 \to$ Laplace).

Γ surface $\subseteq \mathbb{R}^d$

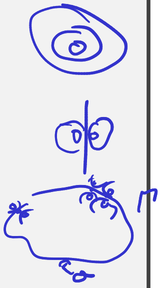
$$S_k\sigma(x) = \int_\Gamma G_k(x-y)\,\sigma(y)\,dS_y$$

$$D_k\sigma(x) = \int_\Gamma \partial_{n_y} G_k(x-y)\,\sigma(y)\,dS_y$$

$$S'_k\sigma(x) = \partial_{n_x}\int G_k(x-y)\,\sigma(y)\,dS_y$$

$$D'_k\sigma(x) = \partial_{n_x}\int_\Gamma \partial_{n_y} G_k(x-y)\,\sigma(y)\,dS_y$$

"the hypersingular op."



These operators map function σ on Γ to...

onto Γ

onto $\mathbb{R}^d$

Layer Potentials (II)

Called *layer potentials*:

- S is called the *single-layer potential*
- D is called the *double-layer potential*
- S'' (and higher) analogously

(Show pictures using `pytential/examples/layerpot.py`, observe continuity properties.)

Alternate (“standard”) nomenclature:

How does this actually solve a PDE?

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u|_\Gamma = f|_\Gamma.$$

"Representation" $u(x) = S_0\sigma(x) = \int_\Gamma G_0(x-y)\,\sigma(y)\,dS_y$

$\lim_{x\to\Gamma^-} S_0\sigma(x) = S_0\sigma(x_0) \quad (x_0 \in \Gamma)$

$\int_\Gamma G_0(x_0-y)\,\sigma(y)\,dS_y = f(x_0)$

unknown density σ

S is compact $\Rightarrow$ IE has many near-nullspaces $\Rightarrow$ bad

IE BVP Solve: Observations (I)

Observations:

- One can choose representations relatively freely. Only constraints:
  - Can I get to the solution with this representation?
    I.e. is the solution I'm looking for represented?
  - Is the resulting integral equation solvable?

Q: How would we know?

IE BVP Solve: Observations (II)

- Some representations lead to better integral equations than others.
  The one above is actually terrible (both theoretically and practically).
  Fix above: Use $u(x) = D\sigma(x)$ instead of $u(x) = S\sigma(x)$.
  Q: How do you tell a good representation from a bad one?
- Need to actually *evaluate* $S\sigma(x)$ or $D\sigma(x)$...
  Q: How?

$\rightarrow$ Need some theory

Outline

Outline

Norms

$L^2(\Omega) := \quad \|f\|_{L} = \sqrt{\int_\Omega |f(x)|^2 \, dx}$

Definition

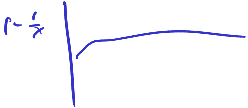
(Norm) A *norm* $\|\cdot\|$ maps an element of a *vector space* into $[0, \infty)$. It satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Can create norm from *inner product*: $\|x\| = \sqrt{\langle x, x \rangle}$

$(f, g) = \int_\Omega f(x) \, g(x) \, dx$

Function Spaces



Name some function spaces with their norms.

- $L^2 \to \sqrt{\int |f|^2}$ — $\ell^2 \to \|\vec{x}\|_2 = \sqrt{\sum x_i^2}$
- L^3 — ℓ^3
- $L^\infty \to \operatorname{ess\,sup}_x |f(x)|$ — $\ell^\infty \to \|\vec{x}\|_\infty = \max |x_i|$
- $C^0(\Omega)$ — Ω closed & bdd. — $\|f\|_\infty = \max_x |f(x)|$
- $C^1(\Omega)$ — $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$

Convergence

Name some ways in which a sequence can 'converge'.

Norm convergence:

$x_n \to x \;:\Leftrightarrow\; \|x_n - x\| \to 0$

↑ need limit

Cauchy sequence:

For all $\varepsilon > 0$ there exists an n so that $\|x_\nu - x_\mu\| < \varepsilon$. $(\nu, \mu \geq n)$

Want: Cauchy sequences have limits ("complete", "Banach")

Operators

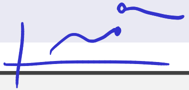
X, Y: Banach spaces, $A : X \to Y$ linear operator

Definition (Operator norm)

$$\|A\| := \sup\{\|Ax\| : x \in X, \|x\| = 1\}$$

Theorem

$\|A\|$ *bounded* $\Leftrightarrow$ *A continuous* ~~(at x?)~~



cont: $x_n \to x \Rightarrow Ax_n \to Ax \Leftarrow A(\underbrace{x_n - x}_{\to 0}) \to A(x-x) = 0$

equiv: $x_n \to 0 \Rightarrow Ax_n \to 0$

$$\|Ax_n\| \le \underbrace{\|A\|}_{<\infty}\,\underbrace{\|x_n\|}_{\to 0} \to 0$$

- What does 'linear' mean here? linear $A(\alpha x + y) = \alpha Ax + Ay$
- Is there a notion of 'continuous at x' for linear operators? no

Operators: Examples

Which of these is bounded as an operator on functions on the real line?

- Multiplication by a scalar ✓
- "Left shift" $f \mapsto f(x+\alpha)$
- Fourier transform L^1 ✓ L^2 ✓ (not obvious)
- Differentiation $e^{\alpha x}$ $\partial_x(e^{\alpha x}) = \alpha e^{\alpha x}$ $\partial_x : C^1 \to C^0$ yes
- Integration of fn yes
- Integral operators ?

depends on the norm.

Integral Equations: Zoology

Volterra	Fredholm
$\int_a^x k(x,y)f(y)dy = g(x)$	$\int_G k(x,y)f(y)dy = g(x)$

First kind	Second Kind ("happy")
$\int_G k(x,y)f(y)dy = g(x)$	$f(x) + \int_G k(x,y)f(y)dy = g(x)$

"sad"/many near nullspaces

$(I+A)f = g$

So $\sigma = f$

Questions:

- First row: First or second kind?
- Second row: Volterra or Fredholm?
- Matrix (i.e. finite-dimensional) analogs?
- What can happen in 2D/3D?
- Factor allowable in front of the identity?
  $(\alpha I + A)f = g$
- Why even talk about 'second-kind operators'?
  - Throw a $+\delta(x-y)$ into the kernel, back to looking like first kind. So?
  - Is the identity in $(I+K)$ crucial?

"second kind"

$$y(x) = f(x) + \int k(x,y)\, f(y)\, dy$$

$$\tilde{k}(x,y) = k(x,y) + \delta(x-y)$$

$$= \int \tilde{k}(x,y)\, f(y)\, dy$$

"First kind"

"second kind": Int. op. compact.

Integral Operators: Boundedness (=Continuity)

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ $\quad x, y \in G$

$G^2 = G \times G$ $\quad (x, y)$

Theorem (Continuous kernel ⇒ bounded)

$G \subset \mathbb{R}^n$ closed, bounded ("compact"), $K \in C(G^2)$. Let

$$(A\phi)(x) := \int_G K(x, y)\phi(y)dy.$$

Then

$$\|A\|_\infty = \max_{x \in G} \int_G |K(x, y)|dy.$$

$|a+b| \leq |a| + |b|$

$|\sum a| \leq \sum |a|$

$|\int a| \leq \int |a|$

Show '$\leqslant$'.

$$\|A\varphi\|_\infty = \max_{x \in G} \left| \int K(x,y)\varphi(y)dy \right|$$

$$\leq \max_{x \in G} \int |K(x,y)| \underbrace{|\varphi(y)|}_{\leq \|\varphi\|_\infty} dy$$

$$\leq \|A\|_\infty \|\varphi\|_\infty$$

Solving Integral Equations

Given

$$(A\phi)(x) := \int_G K(x,y)\varphi(y)dy,$$

are we allowed to ask for a solution of

$$(\mathrm{Id} + A)\varphi = g?$$

$$\frac{1}{1-a} = \sum_{k=0}^{\infty} a^k$$

$$\varphi = (1-A)^{-1} g = \sum_{k=0}^{\infty} A^k g$$

Neumann series

Attempt 1: The Neumann series

Want to solve

$$\varphi - A\varphi = (I - A)\varphi = g.$$

Formally:

$$\varphi = (I - A)^{-1} g.$$

What does that remind you of?