Ann

converge on proj. proposals

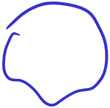
Goals

- Neumann series
- Compactness
  - what?
  - sets of functions are compact.
  - int. operators compact?
  - Riesz theory

Review

- Norms
- Operators / their norm
- boundedness of lin. operators $\Leftrightarrow$ continuity
- limits.
  - convergence
  - limits out of thin air: Cauchy sequences

$G \subset \mathbb{R}^n$



G "compact" $\Leftrightarrow$ closed, bounded

Solving Integral Equations

Given

$$(A\phi)(x) := \int_G K(x,y)\varphi(y)dy,$$

are we allowed to ask for a solution of

$$(\mathrm{Id} - A)\varphi = g?$$

- Neuman
- Riesz (based on compactness)
- Fredholm (— " —)

Attempt 1: The Neumann series

Want to solve

$$\varphi - A\varphi = (I - A)\varphi = g.$$

Formally:

$$\varphi = (I - A)^{-1} g. \quad \text{"}{=}\text{"} \quad \frac{1}{I - A}$$

What does that remind you of?

$$\frac{1}{1-\alpha} = \sum_{k=0}^{\infty} \frac{1}{1-\alpha} \qquad (|\alpha| < 1)$$

Attempt 1: The Neumann series (II)

Theorem

$A : X \to X$ *Banach*, $\|A\| < 1$ $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$ *with* $\|(I - A)^{-1}\| \le 1/(1 - \|A\|)$.

- How does this rely on completeness/Banach-ness?
- There's an iterative procedure hidden in this.
  (Called *Picard Iteration*. Cf: Picard-Lindelöf theorem.)
  Hint: How would you compute $\sum_k A^k f$?

$\|A^0\|$ $\|A^1\|$ $\|A^2\|$ $\|A^3\|$

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.

Q: Why does this fall short?

$$\sum_k A^k f$$

"dumb" way

$x_0 := f$

$x_1 := x_0 + Af$

$x_2 := x_1 + A^2 f$

$x_3 := x_2 + A^3 f$

Picard iteration

$y_0 := f$

$y_1 := f + A y_0$

$y_2 := f + A y_1$

$y_3 := f + A y_2$

Outline

Compact Sets

Definition (Precompact/Relatively compact)

$M \subseteq X$ precompact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in X

Definition (Compact/'Sequentially complete')

$M \subseteq X$ compact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in M

- Precompact $\Rightarrow$ bounded
- Precompact $\Leftrightarrow$ bounded (finite dim. only!)

"compact": precompact + closed

$M = \{ x_0, x_1 \}$

Let $(y_k) \subseteq M$. $y_k := x_{n(k)}$ $\quad n(k) \in \{0, 1\}$

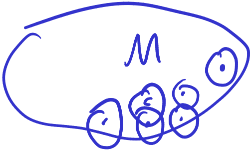
Then. $N(i) = \#\{ n(k) = i : k \in \mathbb{N}_0 \}$ $i \in \{0, 1\}$

$N(0) < \infty \wedge N(1) < \infty$ cannot happen.

Pick $i \in \mathbb{N}_0$ so that $N(i) = \infty$.

Define $m(k) =$ "kth time that x_i is visited by y_k"

$y_{m(k)} = x_i$



"cover":

$$M \subseteq \bigcup_{S \in C} S$$

$S \in C$ are open

overlap is OK

C infinite OK

C uncountably infinite OK

M compact

$\iff$ every cover has a finite subcover F so that

$$M \subseteq \bigcup_{S \in F \subseteq C} S$$

Compact Sets (II)

$$\|x\|_\infty := \sup_{i \in \mathbb{N}_0} |x_i|$$

Counterexample to 'precompact ⇔ bounded'? (∞ dim)

$$x_0 := (1, 0, 0, 0, \ldots)$$
$$x_1 := (0, 1, 0, 0, 0, \ldots)$$
$$x_2 := (0, 0, 1, 0, \ldots)$$

$$\|x_0\|_\infty = 1$$
$$\vdots$$

$$\|x_i - x_j\| = 1 \quad \text{if } i \neq j$$

Compact Operators

X, Y: Banach spaces

Definition (Compact operator)

$T : X \to Y$ is *compact* $:\Leftrightarrow$ T(bounded set) is precompact.

Theorem

- T, S *compact* $\Rightarrow \alpha T + \beta S$ *compact*
- *One of* T, S *compact* $\Rightarrow S \circ T$ *compact*
- T_n *all compact,* $T_n \to T$ *in operator norm* $\Rightarrow T$ *compact*

$(S \circ T)(x) = S(T(x))$

Questions:

- Let $\dim T(X) < \infty$. Is T compact?
- Is the identity operator compact?

yes

if $\dim(Y) < \infty$.

Intuition about Compact Operators

- Compact operator: As finite-dimensional as you're going to get in infinite dimensions.
- Not clear yet–but they are moral (∞-dim) equivalent of a matrix having *low numerical rank*.
- Are compact operators continuous (=bounded)? yes
- What do they do to high-frequency data?
- What do they do to low-frequency data?

must be smoothing

Arzelà-Ascoli

Let $G \subset \mathbb{R}^n$ be compact.

Theorem (Arzelà-Ascoli [Kress LIE 3rd ed. Thm. 1.18]**)**

$U \subset C(G)$ is precompact iff it is bounded and equicontinuous.

Equicontinuous means

for all $f \in U$

For all $\varepsilon > 0$ there exists $\delta > 0$ s.t.
For all x, y: $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Continuous means:

For all $\varepsilon > 0$ there exists $\delta > 0$ s.t.
For all x, y: $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Arzelà-Ascoli: Proof Sketch for b$\wedge$ e $\Rightarrow$ c