April 30, 2024

Announcements

- Pres. day

Goals

- prove Calderón
- discretization
  - ↳ Galerkin
  - ↳ Nyström

Review

- CFIE

Dirichlet: ✓ $D - iS$

Neumann: $S - iD$

ext. Neumann:

$$S'\varphi - iD'\varphi - \frac{\varphi}{2}$$

Fix repr. $S - iDS$

$$S'\varphi - \frac{\varphi}{2} - iD'S\varphi$$

$$D'S = D^2 - \frac{1}{4}$$

Towards Calderón

Show that $(S\varphi, D'\psi) = ((S' + I/2)\varphi, (D - I/2)\psi)$. [Kress LIE 3rd ed. Sec 7.6]

$w := S\varphi \qquad v := D\psi$

$(S\varphi, D'\psi)_{\partial\Omega} = (w, \partial_n v)_{\partial\Omega} = (\partial_n w^-, v^-)_{\partial\Omega} = ((S' + \frac{I}{2})\varphi, (D - \frac{I}{2})\psi)$

$(\varphi, SD'\psi)$?

$(\varphi, SD'\psi) = (S\varphi, D'\psi)$

$= ((S' + \frac{I}{2})\varphi, (D - \frac{I}{2})\psi)$

$= (\varphi, (D + \frac{I}{2})(D - \frac{I}{2})\psi)$

$= (\varphi, (D^2 - \frac{I}{4})\psi)$

Calderón Identities: Summary

- $SD' = D^2 - I/4$ ✓
- $D'S = S'^2 - I/4$

Also valid for Laplace (jump relation same after all!)

Why do we care?

Cfl t Neumann

Outline

Outline

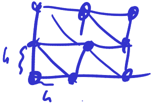
Numerics: What do we need?

- Discretize curves and surfaces
  - Interpolation
  - Grid management
  - Adaptivity
- Discretize densities
- Discretize integral equations
  - Nyström, Collocation, Galerkin
- Compute integrals on them
  - "Smooth" quadrature
  - Singular quadrature
- Solve linear systems

Why high order?

Order p: Error bounded as $|u_h - u| \leq Ch^p$

Thought experiment:



First order	Fifth order
1,000 DoFs ≈ 1,000 triangles	1,000 DoFs ≈ 66 triangles
Error: 0.1	Error: 0.1
Error: 0.01 → ?	Error: 0.01 → ?

Complete the table.

$E(h) \simeq C \cdot h$ | $E(h) \simeq C \cdot h^5$

100x the triangles = 100h | < 4x

Remarks:

- Want $p \geq 3$ available.
- Assumption: Solution sufficiently smooth
- Ideally: p chosen by user

Outline

Integral Equation Discretizations: Overview

$$\phi(x) - \int_\Gamma K(x,y)\phi(y)dy = f(y)$$

Nyström	Projection
▶ Approximate integral by quadrature: $\int_\Gamma f(y)dy \to \sum_{k=1}^n \omega_k f(y_k)$ ▶ Evaluate quadrature'd IE at quadrature nodes, solve	▶ Consider residual: $R := \phi - A\phi - f$ ▶ Pick projection P_n onto finite-dimensional subspace $P_n\phi := \sum_{k=1}^n \langle\phi, v_k\rangle w_k \quad \to$ DOFs $\langle\phi, v_k\rangle$ ▶ Solve $P_n R = 0$

Projection/Galerkin

- Equivalent to projection: Test IE with test functions
- Important in projection methods: *sub*-space (e.g. of $C(\Gamma)$)

Name some generic discrete projection bases.

- Galerkin (test = trial)
- Collocation (test = δ)
- Petrov-Galerkin (test ≠ trial)

Collocation and Nyström: the same?

no

Are projection methods implementable?

not honestly

Outline

Nyström Discretizations (1/4)

Nyström consists of two distinct steps:

1. Approximate integral by quadrature:

$$\varphi_n(x) - \sum_{k=1}^{n} \omega_k K(x, y_k)\varphi_n(y_k) = f(x) \qquad (1)$$

2. Evaluate quadrature'd IE at quadrature nodes, solve discrete system

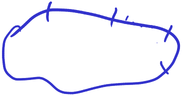
$$\varphi_j^{(n)} - \sum_{k=1}^{n} \omega_k K(x_j, y_k)\varphi_k^{(n)} = f(x_j) \qquad (2)$$

with $x_j = y_j$ and $\varphi_j^{(n)} = \varphi_n(x_j) = \varphi_n(y_j)$

Is version (1) solvable?

no, too many rows

Nyström Discretizations (2/4)



What's special about (2)?

Solvable ✓   no cont. density: ✗

Solution density also only known at point values. But: can get approximate continuous density. How?

$\varphi - A\varphi = f \Leftrightarrow \varphi(x) = f(x) + A\varphi(x)$

↑ Nyström interpolation

Assuming the IE comes from a BVP. Do we also only get the BVP solution at discrete points?

can eval at any target.

Nyström Discretizations (3/4)

Does (1) $\Rightarrow$ (2) hold?

✓

Does (2) $\Rightarrow$ (1) hold?

with φ_n = the Nyström interpolant, yes!

Nyström Discretizations (4/4)

What good does that do us?

THEORY!

Does Nyström work for first-kind IEs?

$\varphi - A\varphi = f$ no.

Convergence for Nyström (1/2)

Increase number of quadrature points n:
Get sequence (A_n)
Want $A_n \to A$ in some sense
What senses of convergence are there for sequences of functions f_n?

- pointwise
- uniform w.r.t. $\|\cdot\|_\infty$
- a few others

$\|f_n - f\|_\infty \not\to 0$

$f_n(x) = x^n$ $[0,1]$

What senses of convergence are there for sequences of operators A_n?

- "pointwise" $\|A_n\varphi - A\varphi\| \to 0$ $\forall \varphi$
- uniform $\|A_n - A\|_\infty \to 0$

Convergence for Nyström (2/2)

Will we get norm convergence $\|A_n - A\|_\infty \to 0$ for Nyström? [Kress LIE 3rd ed. Thm. 12.8]

$\psi_\varepsilon = 1$ near each quad point, 0 otherwise

- $\|A_n \varphi \psi_\varepsilon - A \varphi \psi_\varepsilon\| \to 0$
- $\|A_n - A\|_\infty \geq \|A\|_\infty$

Is functionwise convergence good enough?

no:

Compactness-Based Convergence

X Banach space (think: of functions)

Theorem (Not-quite-norm convergence [Kress LIE 2nd ed. Cor 10.4])

$A_n : X \to X$ bounded linear operators,
functionwise convergent to $A : X \to X$
Then convergence is uniform on compact subsets $U \subset X$, i.e.

$$\sup_{\phi \in U} \|A_n\phi - A\phi\| \to 0 \qquad (n \to \infty)$$

How is this different from norm convergence?

U cannot be the unit ball.

Collective Compactness

Set $\mathcal{A}$ of operators $A : X \to X$

Definition (Collectively compact)

$\mathcal{A}$ is called *collectively compact* if and only if
for $U \subset X$ bounded, $\mathcal{A}(U)$ is relatively compact.

What was relative compactness (=precompactness)?

Collective Compactness: Questions (1/2)

Is each operator in the set $\mathcal{A}$ compact?

Is collective compactness the same as "every operator in $\mathcal{A}$ is compact"?

Collective Compactness: Questions (2/2)

When is a sequence collectively compact?

Is the limit operator of such a sequence compact?

How can we use the two together?

Making use of Collective Compactness

X Banach space, $A_n : X \to X$, (A_n) collectively compact, $A_n \to A$ functionwise.

Corollary (Post-compact convergence [Kress LIE 3rd ed. Cor 10.11]**)**

- $\|(A_n - A)A\| \to 0$
- $\|(A_n - A)A_n\| \to 0$
  $(n \to \infty)$

Anselone's Theorem

$(I-A)\varphi = f$

$(I-A)^{-1}$ exists, with $A : X \to X$ compact, $(A_n) : X \to X$ collectively compact and $A_n \to A$ functionwise.

Theorem (Nyström error estimate [Kress LIE 3rd ed. Thm 10.12])

For sufficiently large n, $(I - A_n)$ is invertible and

$$\|\phi_n - \phi\| \le C(\|(A_n - A)\phi\| + \|f_n - f\|)$$

$$C = \frac{1 + \|(I-A)^{-1}A_n\|}{1 - \|(I-A)^{-1}(A_n - A)A_n\|}$$

$I + (I-A)^{-1}A = ?$ $\quad (I-A)^{-1}$

$$1 + \frac{a}{1-a} = \frac{1-a}{1-a} + \frac{a}{1-a} = \frac{1}{1-a}$$

Anselone's Theorem: Proof (I)

Define approximate inverse $B_n = I + (I - A)^{-1}A_n$.

How good of an inverse is it?

$$
\begin{aligned}
\mathrm{Id} &\approx^? B_n(I - A_n) \\
&= (I + (I - A)^{-1}A_n)(I - A_n) \\
&= [I + (I - A)^{-1}A_n] - [A_n + (I - A)^{-1}A_nA_n] \\
&= [I + (I - A)^{-1}A_n] - [(I - A)^{-1}(I - A)A_n + (I - A)^{-1}A_nA_n] \\
&= [I + (I - A)^{-1}A_n] - [(I - A)^{-1}IA_n - (I - A)^{-1}AA_n + (I - A)^{-1}A_nA_n] \\
&= I + (I - A)^{-1}\,\mathrm{AA}_n - (I - A)^{-1}A_nA_n \\
&= I + \underbrace{(I - A)^{-1}(A - A_n)A_n}_{-S_n} = I - S_n
\end{aligned}
$$

Anselone's Theorem: Proof (II)

Want $S_n \to 0$ somehow. Prior result gives us $\|(A - A_n)A_n\| \to 0$.

$\|S_n\| \to 0$

$(1 - S_n)^{-1}$ exists....

$B_n(1 - A_n) = 1 - S_n$

$(1 - A_n)^{-1} = (1 - S_n)^{-1} B_n$

exists!

Anselone's Theorem: Proof (III)