April 18, 2024

Announcements

- HW4 due

Goals

- rev. jump rels
- BVP uniqueness
- draft of repr. choice
- show IE uniqueness for all 4 BVPs
  w/ patch ext. Dirichlet
- corners?
- Helmholtz

Review

- Jump relations
- BVP uniqueness
- $\Delta u = 0$

$$u(x) = S\sigma(x)$$

Jump Relations: Mathematical Statement

Let $[X] = X_+ - X_-$. (Normal points towards "+"="exterior".) Let $x_0 \in \Gamma$.

Theorem (Jump Relations [Kress LIE 3rd ed. Thm. 6.15, 6.18,6.19]**)**

$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I\right)(\sigma)(x_0) \quad \Rightarrow \quad [S'\sigma] = -\sigma$$

$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I\right)(\sigma)(x_0) \quad \Rightarrow \quad [D\sigma] = \sigma$$

$$[D'\sigma] = 0$$

Truth in advertising: Assumptions on Γ?

$\Gamma \in C^2$

Representations for BVP solutions (Draft)

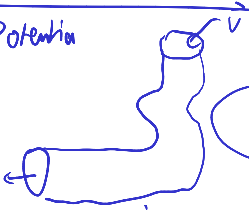
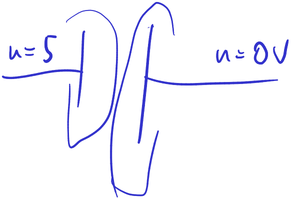
Requirements:
- $\Delta u = 0$ on Ω
- SKIE
- SKIE uniqueness

	Dirichlet	Neumann
int.	$\lim_{x \to \partial\Omega^-} u(x) = g(x)$ ✓ $u(x) = D\sigma(x)$ $\lim_{x \to \partial\Omega^-} u(x) = D\sigma - \frac{\sigma}{2} = g$ SKIE $u(x) = S\sigma(x)$ $\lim_{x \to \partial\Omega^-} S\sigma(x) = S\sigma(x) = g(x)$ (on surf.) ? (not 2nd kind	$\lim_{x \to \partial\Omega^-} \partial_n u = g(x)$ $u(x) = S\sigma(x)$ $\lim_{x \to \partial\Omega^-} \partial_n u(x) = S'\sigma + \frac{\sigma}{2} = g$
ext.	$\lim_{x \to \partial\Omega^+} u(x) = g(x)$ $u(x) = D\sigma(x)$ $\lim_{x \to \partial\Omega^+} u(x) = D\sigma + \frac{\sigma}{2} = g$ SKIE	$\lim_{x \to \partial\Omega^+} \partial_n u = g(x)$ ✓ $u(x) = S\sigma(x)$ $\lim_{x \to \partial\Omega} \partial_n u(x) = S'\sigma - \frac{\sigma}{2} = g$ $N\begin{pmatrix} D \pm \frac{1}{2}I \\ S' \pm \frac{1}{2}I \end{pmatrix} = ?$

$$\Delta u = f$$

$$\tilde{u} = G * f \qquad \Delta \tilde{u} = f$$

$$u = \tilde{u} + u_0 \qquad \Delta u_0 = 0$$

	Dir	Neuman
int.		Potential  $\Delta p = 0$ $v = -\nabla p$ prescribed in/out flow
ext.		

Boundary Value Problems: Overview

	Dirichlet	**Neumann**
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ ⊕ unique	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ O may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\lvert x\rvert \to \infty$ ⊕ unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\lvert x\rvert \to \infty$ ⊕ unique

with $g \in C(\partial\Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

Uniqueness Proofs

$\Delta u = 0$

$u|_{\partial\Omega} = 0 \Rightarrow u = 0.$

Dirichlet uniqueness: why?

Suppose u_1, u_2 both solve (int.) Dir. BVP. $u = u_1 - u_2$

Neumann uniqueness: why?

$\partial_n u_1 = g$

$\partial_n u_2 = g$

Suppose u_1, u_2 both solve (int.) Neuman BVP. $u = u_1 - u_2$.

$\partial_n u = 0$ on $\partial\Omega$. If u is non-constant,

$\nabla u \neq 0$ somewhere.

$$\int_\Omega u \cancel{\Delta u}^{0} + \nabla u \cdot \nabla u = \int_{\partial\Omega} u \, \partial_n u$$

$$0 < \int_\Omega |\nabla u|^2 = 0$$

Uniqueness: Remaining Points

Truth in advertising: Missing assumptions on Ω?

$\partial\Omega \in C^2$

What's a DtN map? Dirichlet-to-Neumann

Poincaré-Steklov operator

Next mission: Find IE representations for each.

Uniqueness of Integral Equation Solutions

Theorem (Nullspaces [Kress LIE 2nd ed. Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \operatorname{span}\{1\}$, $N(I/2 + S') = \operatorname{span}\{\psi\}$, *where* $\int \psi \neq 0$.

ext. Dirichlet

int. Neumann

IE Uniqueness: Proofs (1/3)

Show $N(I/2 - D) = \{0\}$.

Suppose $\frac{\varphi}{2} - D\varphi = 0$. To show: $\varphi = 0$. $u(x) := D\varphi(x)$

$\lim_{x \to \Gamma_-} u(x) = 0$. $u \equiv 0$ inside Γ. $\partial_n u_- = 0$. Jump rel for D'

says $(\partial_n u)^+ = 0$. So u sat. an ext. Neumann problem

w/ zero bdry data. (specifically, has right decay).

$\Rightarrow u = 0$ outside Γ.

Since $[D\varphi] = \varphi$, $0 = [u] = \varphi$.

IE Uniqueness: Proofs (2/3)

Show $N(I/2 - S') = \{0\}$.

Fred. Holm.

IE Uniqueness: Proofs (3/3)

Show $N(I/2 + D') = \text{span}\{1\}$.

What conditions on the RHS do we get for int. Neumann and ext. Dirichlet?

$\to$ "Clean" Existence for 3 out of 4