

## Announcements

- HW1

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## Goals

- Error estimates for Taylor of potentials
- multipoles
- towards fast algorithms

## Review

$N \times N$

$N^2 k$

$Nk^2$

- 1D, cheap LR, SVDs
- cheap "compressed forms"
- Taylor / smoothness for rank finding

## Taylor and Error (II)

Now suppose that we had an estimate that

$$\left| \frac{f^{(p)}(c)}{p!} h^p \right| \leq \alpha^p.$$

Assume:  $f(c+h) = \sum_{p=0}^{\infty} \frac{f^{(p)}(c)}{p!} h^p$

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$f(c+h) = \sum_{p=0}^k \frac{f^{(p)}(c)}{p!} h^p + \underbrace{\sum_{p=k+1}^{\infty} \frac{f^{(p)}(c)}{p!} h^p}$$

$$\left| f(c+h) - \sum_{p=0}^k \frac{f^{(p)}(c)}{p!} h^p \right| \leq \sum_{p=k+1}^{\infty} \alpha^p \stackrel{|\alpha| < 1}{=} \frac{1}{1-\alpha} \cdot \alpha^{k+1}$$

## Connect Taylor and Low Rank

Can Taylor help us establish low rank of an interaction?

$$\begin{aligned} f(c+h) &\approx \sum_{p=0}^k \frac{f^{(p)}(c)}{p!} h^p \\ &= \sum_{p=0}^k \text{coeff}_p \text{basis}_p(x) \end{aligned}$$

⇒ use this to establish low rank if smooth

# Taylor on Potentials (I)

Compute a Taylor expansion of a 2D Laplace point potential.

$$G(\vec{x}-\vec{y}) = \log \frac{1}{\|\vec{x}-\vec{y}\|_2}$$

Free-space  
GF:  
 $\Delta u = \delta$   
↑  
2D Laplace  
Fundamental  
 $\log(x^2 + y^2)$

$$\begin{aligned}\psi(\vec{x}) &= \sum_{j=1}^N G(\vec{x}, \vec{y}_j) \varphi(\vec{y}_j) \\ &= \sum_{j=1}^N \log \|\vec{x} - \vec{y}_j\|_2 \varphi(\vec{y}_j)\end{aligned}$$

WLOG: consider one source point

$$\psi(\vec{x}) = \log(\|\vec{x} - \vec{y}\|_2)$$

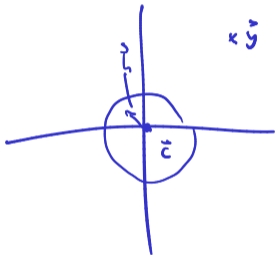
WLOG: pick Taylor exp ctr.  
at  $\vec{0}$ .

$$\Delta \cdot \Delta = \partial_x^2 + \partial_y^2$$

$$\psi(\vec{h}) \approx \sum_{|\vec{p}| \leq k} \leftarrow$$

$$\frac{D^{\vec{p}} \psi(\vec{0})}{\vec{p}!} h^{\vec{p}}$$

subgoals: understand growth of those



$$|\vec{p}|=1 : (0,1) \quad (1,0)$$

$$|\vec{p}|=2 : (2,0) \quad (1,1) \quad (0,2)$$

$$\in \|y\|_{\infty}^5 \leq C \|y\|_2^5$$

$$24 y_2^5 - 240 y_1^2 y_2^3 + 120 y_1^4 y_2^2$$

(%010) -----

$$24 (y_2^{10} + 5 y_1^2 y_2^8 + 10 y_1^4 y_2^6 + 10 y_1^6 y_2^4 + 5 y_1^8 y_2^2 + y_1^{10})$$

$$\} \in \|y\|_2^5$$

$$\sim \|y\|_2^{10}$$

$$\sim \frac{1}{\|y\|_2^5}$$

## Taylor on Potentials (Ia)

Why is it interesting to consider Taylor expansions of Laplace point potentials?

- pushes boundary of "smooth"
- important app

## Taylor on Potentials (II)

Maxima 5.42.1 <http://maxima.sourceforge.net>

(%i1) phi0: log(sqrt(y1\*\*2 + y2\*\*2));

(%o1) 
$$\frac{\log(y_2^2 + y_1^2)}{2}$$

(%i2) diff(phi0, y1);

(%o2) 
$$\frac{y_1}{y_2^2 + y_1^2}$$

(%i3) diff(phi0, y1, 5);

(%o3) 
$$\frac{120 y_1}{(y_2^2 + y_1^2)^3} - \frac{480 y_1}{(y_2^2 + y_1^2)^4} + \frac{384 y_1}{(y_2^2 + y_1^2)^5}$$

(%i4)



## Taylor on Potentials (III)

Which of these is the most dangerous (largest) term?

all have same powers

What's a bound on it? Let  $R = \sqrt{y_1^2 + y_2^2}$ .

$$R = \|\vec{y}\|_2$$

For  $p \geq 1$ :  $\frac{1}{R^p}$

'Generalize' this bound:

## Taylor on Potentials (IV)

What does this mean for the convergence of the Taylor series as a whole?

$$\underbrace{\left| \frac{D^p \psi(0)}{p!} h^p \right|}_{\text{Term } p} \leq C_p \frac{1}{R^{\alpha}} \|h\|^p = C_p \underbrace{\left( \frac{\|h\|}{R} \right)^p}_{\approx \alpha}$$

Err estimate :

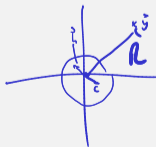
$$\frac{1}{1-\alpha} \alpha^{p+1}$$

if we ignore

$C_p$

Conv. crit:  $|a| < 1$

$$\vec{x} = \vec{c} + \vec{h}$$



$$\alpha = \frac{\|h\|}{R} < 1$$

# Taylor on Potentials (V)

Lesson?

target

Conv. factor  $\frac{\|\vec{x} - \vec{c}\|_2}{\|\vec{y} - \vec{c}\|_2}$

source

The diagram shows a rounded rectangular box containing the handwritten text "Conv. factor" followed by the fraction  $\frac{\|\vec{x} - \vec{c}\|_2}{\|\vec{y} - \vec{c}\|_2}$ . A blue arrow labeled "target" points to the numerator  $\|\vec{x} - \vec{c}\|_2$ , and another blue arrow labeled "source" points to the denominator  $\|\vec{y} - \vec{c}\|_2$ .

## Taylor on Potentials (VI)

Generalize this to multiple source points:

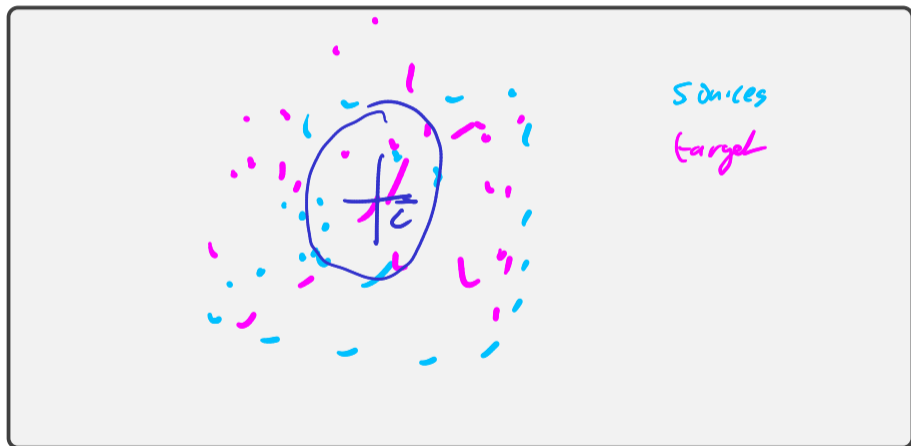
Error bound:

$$\left( \frac{\max_j \|\vec{x}_j - \vec{c}\|_2}{\min_j \|\vec{y}_j - \vec{c}\|_2} \right)^{p+1} - \left( \frac{\text{dist}(\vec{c}, \text{farthest target})}{\text{dist}(\vec{c}, \text{closest source})} \right)^{p+1}$$

one src:  
one tgt:  $\left( \frac{\|\vec{x} - \vec{c}\|_2}{\|\vec{y} - \vec{c}\|_2} \right)^{p+1}$

## Local expansions as a Computational Tool

Low rank makes evaluating interactions cheap(er). Do local expansions help with that goal?



sources



targets



$l$  terms in the expansion

Find coeffs:

Cost:  
 $O(Sk)$

Evaluate expansion:

$\frac{O(Tk)}{O(Sk) + O(Tk)}$

"naive"/direct eval  
 $O(ST)$

# Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

## Rank and Smoothness

Local Expansions

**Multipole Expansions**

Rank Estimates

Proxy Expansions

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

## Taylor on Potentials, Again

Stare at that Taylor formula again.

(single src, single tgt)

Local:  $\psi(\vec{x} - \vec{y}) \approx \sum_{|\vec{p}| \leq k} \underbrace{\frac{\partial^{\vec{p}} \psi(\vec{x} - \vec{y})}{\vec{p}!} \Big|_{\vec{x} = \vec{c}}}_{\text{dep. on src}} \underbrace{(\vec{x} - \vec{c})^{\vec{p}}}_{\text{dep. on tgt}}$

$\vec{x}$  : tgt       $\vec{y}$  : src.

Multipole expn:

$$\psi(\vec{x} - \vec{y}) \approx \sum_{|\vec{p}| \leq k} \underbrace{\frac{\partial^{\vec{p}} \psi(\vec{x} - \vec{y})}{\vec{p}!} \Big|_{\vec{y} = \vec{c}}}_{\text{dep on tgt}} \underbrace{(\vec{y} - \vec{c})^{\vec{p}}}_{\text{dep. on src}}$$



## Multipole Expansions (I)

At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

**First Q:** When does this expansion converge?

