Announcements

- HW1

Goals

- Error estimates for Taylor of potentials
- multipoles
- towards fast algorithms

Review

$N \times N$

$N^2 k \rightarrow Nk^2$

- ID, cheap LR SVDs
- cheap "compressed forms"
- Taylor / smoothness for rank finding

Taylor and Error (II)

Now suppose that we had an estimate that $\left|\dfrac{f^{(p)}(c)}{p!}h^p\right| \leqslant \alpha^p.$

Assume: $f(c+h) = \displaystyle\sum_{p=0}^{\infty} \frac{f^{(p)}(c)}{p!} h^p$

$\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$

$$f(c+h) = \sum_{p=0}^{k} \frac{f^{(p)}(c)}{p!} h^p + \underbrace{\sum_{p=k+1}^{\infty} \frac{f^{(p)}(c)}{p!} h^p}$$

$$\left| f(c+h) - \sum_{p=0}^{k} \frac{f^{(p)}(c)}{p!} h^p \right| \leqslant \sum_{p=k+1}^{\infty} \alpha^p \overset{!}{=} \frac{1}{1-\alpha} \cdot \alpha^{k+1} \qquad |\alpha| < 1$$

Connect Taylor and Low Rank

Can Taylor help us establish low rank of an interaction?

$$f(c+h) \approx \sum_{p=0}^{k} \frac{f^{(p)}(c)}{p!} h^p$$

$$= \sum_{p=0}^{k} \text{coeff}_p \; \text{basis}_p(\underbrace{x}_{c+h})$$

$\Rightarrow$ use this to establish low rank if smooth

Taylor on Potentials (I)

Compute a Taylor expansion of a 2D Laplace point potential.

$G(\vec{x}-\vec{y}) = \log(\|\vec{x}-\vec{y}\|_2)$

Free-space GF:

$\leftrightarrow \Delta u = \delta$

2D Laplace Fundamental

$\log(x^2+y^2)$

$\nabla \cdot \nabla = \partial_x^2 + \partial_y^2$

$$\psi(\vec{x}) = \sum_{j=1}^{n} G(\vec{x}, \vec{y}_j)\,\varphi(\vec{y}_j)$$

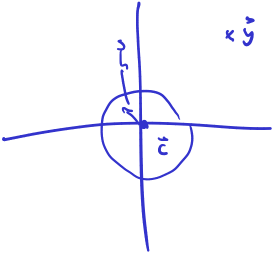
$$= \sum_{j=1}^{n} \log\|\vec{x}-\vec{y}\|_2\,\varphi(\vec{y}_j)$$

WLOG: consider one source point

$$\psi(\vec{x}) = \log(\|\vec{x}-\vec{y}\|_2)$$

WLOG: pick Taylor exp ctr. at $\vec{0}$.

$$\psi(\vec{h}) \approx \sum_{|\vec{p}| \le k} \underbrace{\frac{D^{\vec{p}}\psi(\vec{0})}{\vec{p}\,!}}_{\text{sub goal: understand growth of those}} \vec{h}^{\vec{p}}$$



$|\vec{p}| = 1$: $(0,1)$ $(1,0)$

$|\vec{p}| = 2$: $(2,0)$ $(1,1)$ $(0,2)$

$$\leq C \|y\|_\infty^5 \leq C \|y\|_2^5$$

(%o10) $$\frac{24\, y2^5 - 240\, y1^2\, y2^3 + 120\, y1^4\, y2}{24\, (y2^{10} + 5\, y1^2\, y2^8 + 10\, y1^4\, y2^6 + 10\, y1^6\, y2^4 + 5\, y1^8\, y2^2 + y1^{10})}$$

$$\} \leq \|y\|_2^5$$

$$\sim \|y\|_2^{10} \qquad \sim \frac{1}{\|y\|_2^5}$$

Taylor on Potentials (Ia)

Why is it interesting to consider Taylor expansions of Laplace point potentials?

- pushes boundary of "smooth"
- important app

Taylor on Potentials (II)

```
Maxima 5.42.1 http://maxima.sourceforge.net
(%i1) phi0: log(sqrt(y1**2 + y2**2));
                                   2     2
                              log(y2  + y1 )
(%o1)                         --------------
                                   2
(%i2) diff(phi0, y1);
                                   y1
(%o2)                           ---------
                                  2     2
                                y2  + y1
(%i3) diff(phi0, y1, 5);
                                        3              5
                120 y1        480 y1         384 y1
(%o3)         ------------ - ------------ + ------------
                 2     2 3      2     2 4      2     2 5
              (y2  + y1 )    (y2  + y1 )    (y2  + y1 )
(%i4)
```

Taylor on Potentials (III)

Which of these is the most dangerous (largest) term?

all have same powers

What's a bound on it? Let $R = \sqrt{y_1^2 + y_2^2}$.

$R = \|\vec{y}\|_2$

For $p \geq 1$: $\frac{1}{R^p}$

'Generalize' this bound:

Taylor on Potentials (IV)

What does this mean for the convergence of the Taylor series as a whole?

$$\left|\frac{D^p\psi(0)}{p!}h^p\right| \le C_p\frac{1}{R^p}\|h\|^p = C_p\left(\underbrace{\frac{\|\vec{h}\|}{R}}_{\approx\alpha}\right)^p$$

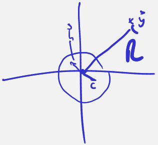
if we ignore C_p

Err estimate: $\frac{1}{1-\alpha}\alpha^{p+1}$

① ②

Conv. crit: $|\alpha|<1$

$\vec{x}=\vec{c}+\vec{h}$



$$\alpha=\frac{\|\vec{h}\|}{R}<1$$

Taylor on Potentials (V)

Lesson?

target

Conv. factor $\dfrac{\|\vec{x}-\vec{c}\|_2}{\|\vec{y}-\vec{c}\|_2}$

source

Taylor on Potentials (VI)

Generalize this to multiple source points:

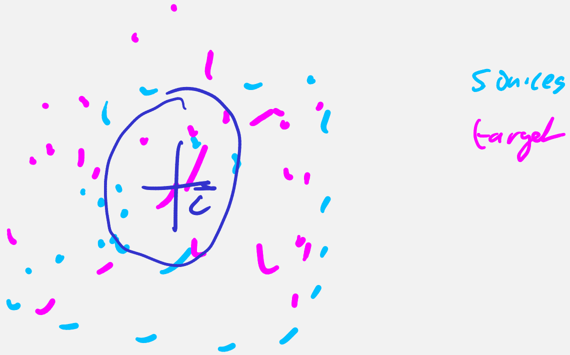
Error bound:

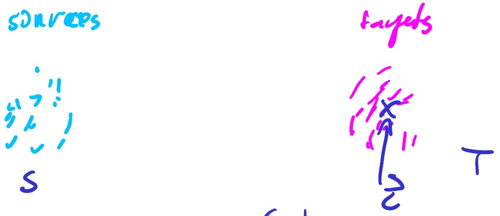
One src: one tgt: $\left(\frac{\|\vec{x}-\vec{c}\|_2}{\|\vec{y}-\vec{c}\|_2}\right)^{p+1}$

$$\left(\frac{\max_i \|\vec{x}_i-\vec{c}\|_2}{\min_j \|\vec{y}_j-\vec{c}\|_2}\right)^{p+1} = \left(\frac{\text{dist}(\vec{c}, \text{farthest target})}{\text{dist}(\vec{c}, \text{closest source})}\right)^{p+1}$$

Local expansions as a Computational Tool

Low rank makes evaluating interactions cheap(er). Do local expansions help with that goal?





k terms in the expansion

Cost:

Find coeffs: $O(Sk)$

Evaluate expansion: $+\ O(Tk)$

$O(Sk) + O(Tk)$

"naive"/direct eval

$O(ST)$

Outline

Taylor on Potentials, Again

Stare at that Taylor formula again. (single src, single tgt)

Local:

$$\psi(\vec{x}-\vec{y}) \approx \sum_{|\vec{p}|\le k} \underbrace{\frac{D^{\vec{p}}\psi(\vec{x}-\vec{y})\big|_{\vec{x}=\vec{c}}}{\vec{p}!}}_{\text{dep. on src}} \underbrace{(\vec{x}-\vec{c})^{\vec{p}}}_{\text{dep. on tgt.}}$$

$\vec{x}$: tgt $\qquad \vec{y}$: src.

Multipole expn:

$$\psi(\vec{x}-\vec{y}) \approx \sum_{|\vec{p}|\le k} \underbrace{\frac{D^{\vec{p}}\psi(\vec{x}-\vec{y})\big|_{\vec{y}=\vec{c}}}{\vec{p}!}}_{\text{dep on tgt.}} \underbrace{(\vec{y}-\vec{c})^{\vec{p}}}_{\text{dep. on src}}$$

Multipole Expansions (I)

At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

First Q: When does this expansion converge?