Ann

- no office hours today
- project proposals
- HW 4
- proj. presentations
  May 7, May 9
  proj. mat' due May 10

Goals

- Fredholm
- spectral theory
- PDEs.
  ↳ uniqueness.

Review

A compact second-kind

$L = I - A$

R1 $\dim N(L) < \infty$

$L\varphi = g$

R4 L injective $\iff$ L surjective

uniqueness existence

important: "$\Rightarrow$"

R4 totally falls apart if

$N(L) = \{0\}$.

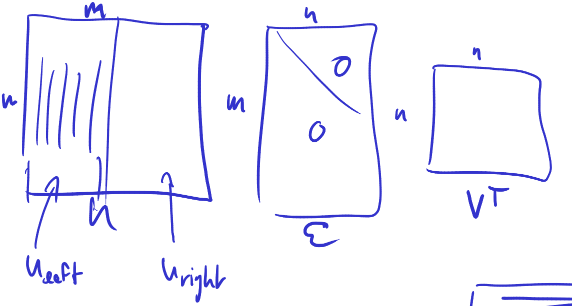
$A\vec{x} = \vec{b}$ $A \in \mathbb{R}^{m\times n}$

$A\vec{x} = \vec{b}$

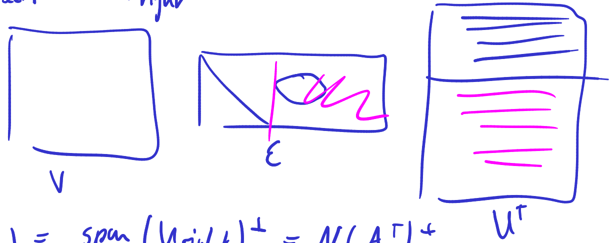
$A \in \mathbb{R}^{m \times n}$

$m > n$

$A =$



$A^T =$



$A(\mathbb{R}^n) = \text{span}(U_{left}) = \text{span}(U_{right})^\perp = N(A^T)^\perp$

Continuous and Square-Integrable

Can we carry over $C^0(G)$ boundedness/compactness results to $L^2(G)$?

X, Y normed spaces with a scalar product so that $|(\phi,\psi)| \le \|\phi\|\,\|\psi\|$ for $\phi,\psi \in X$.

Theorem (Lax dual system [Kress LIE 3rd ed. Thm. 4.13])

Let $U \subseteq X$ be a subspace and let $A : X \to Y$ and $B : Y \to X$ be bounded linear operators with

$$(A\phi,\psi) = (\phi, B\psi) \qquad (\phi \in U, \psi \in Y).$$

Then $A : U \to Y$ is bounded with respect to $\|\cdot\|_s$ induced by the scalar product and $\|A\|_s^2 \le \|A\|\,\|B\|$.

Based on this, it is also possible to carry over compactness results.

from C^0 to L^2.

Adjoint Operators

Definition (Adjoint oeprator)

A^* called adjoint to A if

$(x, y) = x^T y$

$$(Ax, y) = (x, A^* y)$$

$(Ax, y) = (Ax)^T y = (x, A^T y)$

for all x, y.

Facts:

- A^* unique
- A^* exists
- A^* linear
- A bounded $\Rightarrow$ A^* bounded
- A compact $\Rightarrow$ A^* compact

Adjoint Operator: Observations?

What is the adjoint operator in finite dimensions? (in matrix representation)

$$A^T$$

What do you expect to happen with integral operators?

$$A\varphi(x) = \int K(x,y)\,\varphi(y)\,dy \qquad A^*\varphi(x) = \int K(y,x)\,\varphi(y)\,dy$$

Adjoint of the single-layer?

Laplace 2D $S\varphi(x) = c\int \log(|x-y|_2)\,\varphi(y)\,dy$ $\qquad S^* = S$

self-adjoint ("symmetric")

Adjoint of the double-layer?

$$D\varphi(x) = c\int \partial_{n_y} \log(|x-y|_2)\,\varphi(y)\,dy \qquad D^* = S'$$

$$S'\varphi(x) = c\int \partial_{n_x} \log(|x-y|_2)\,\varphi(y)\,dy$$

Fredholm Alternative

Theorem (Fredholm Alternative [Kress LIE 3rd ed. Thm. 4.17])

$A : X \to X$ *compact.* *Then either:*

- $I - A$ *and* $I - A^*$ *are bijective*

  or:

- $\dim N(I - A) = \dim N(I - A^*)$ $< \infty$
- $(I - A)(X) = N(I - A^*)^\perp$
- $(I - A^*)(X) = N(I - A)^\perp$

coincide w/ Riesz.

$\left(\frac{1}{2} \pm D\right)\varphi = g$

$g \perp 1$

$(I - A)\varphi = g$

solvable iff

$g \perp N(I - A^*)$

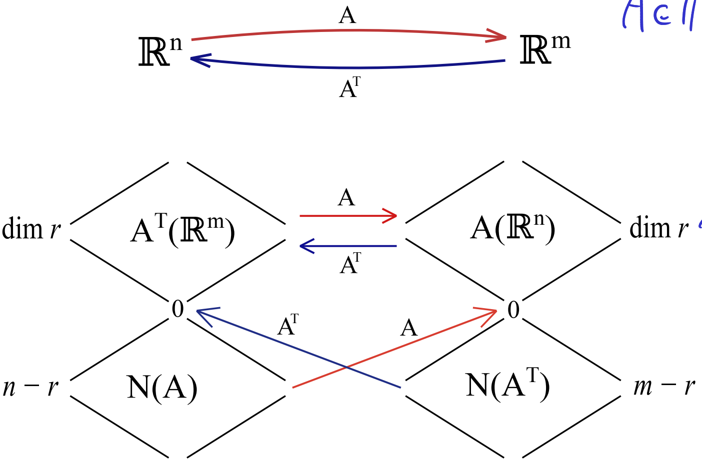
Seen these statements before?

$\boxed{\text{FTLA}}$

$\operatorname{span}\{z\} = N(I - A^*)$ $\quad\{f \in \mathbb{C}^n : f \perp z\}$

Fundamental Theorem of Linear Algebra

$A \in \mathbb{R}^{m \times n}$



rank

[Credit: Wikipedia]

Fredholm Alternative in IE terms

Translate to language of integral equation solvability:

(see above: RHS orth. to $N(A^*)$)

Fredholm Alternative: Further Thoughts

What about symmetric kernels ($K(x, y) = K(y, x)$)?

$A = A^*$

Where to get uniqueness? / statement about $N(I-A^*)$

From elsewhere (PDE)

Outline

Spectral Theory: Terminology

$Ax=\lambda x \iff (A-\lambda I)x=0$

$N(A-\lambda I)\neq\{0\}$

$A : X \to X$ bounded, λ is a … value:

Definition (Eigenvalue)

There exists an element $\phi \in X$, $\phi \neq 0$ with $A\phi = \lambda\phi$.

Definition (Regular value)

The "resolvent" $(\lambda I - A)^{-1}$ exists and is bounded. "not eigen"

Can a value be regular and "eigen" at the same time?

no

What's special about ∞-dim here?

allowed { injective, but not surjective; surjective, but not injective.

∞-dim only

For λ	reg	eign	
	F	F	✓
	F	T	✓
	T	F	✓
	~~T~~	~~T~~	

ℓ^∞: $a = (a_0, a_1, \ldots)$

$\downarrow$

A: $Aa = (0, a_0, a_1, \ldots)$ $\Leftarrow$ $A: \ell^\infty \to \ell^\infty$

injective (no null space)

$\forall a, b \in \ell^\infty$:

not surjective

$(1, 0, 0, 0, \ldots) \in \ell^\infty$

$\nexists \varphi : A\varphi =$

Resolvent Set and Spectrum

Definition (Resolvent set)

$\rho(A) := \{\lambda \text{ is regular}\}$

Definition (Spectrum)

$\sigma(A) := \mathbb{C} \setminus \rho(A)$

Spectral Theory of Compact Operators

Theorem

$A : X \to X$ *compact linear operator, X ∞-dim.*

Then:

- $0 \in \sigma(A)$ $\quad$ 0 is not a reg. value
- $\sigma(A) \setminus \{0\}$ *consists only of eigenvalues*
- $\sigma(A) \setminus \{0\}$ *is at most countable*
- $\sigma(A)$ *has no accumulation point except for 0*

Spectral Theory of Compact Operators: Proofs

Show the first part.

Show second part.