Ann:

- HW4

Goals

- harmonic f. <
- jump rel.
- BVP uniqueness
- IE uniqueness.

Review

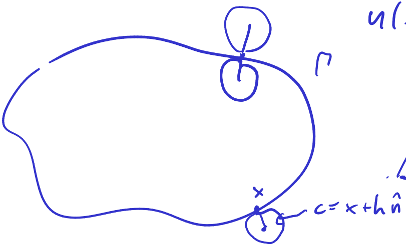
$\Delta u = f \leftarrow$ Poisson

$\Delta u = 0 \leftarrow$ Laplace

- Harmonic functions
- Overall goal: existence for IE.

  R/F: uniqueness $\Rightarrow$ existence

  $\uparrow$ { BVP uniqueness, jump relations }



Taylor expn

$$u(x) = S\sigma(x) = \int_\Gamma \log|x-y|\,\sigma(y)\,dS_y$$

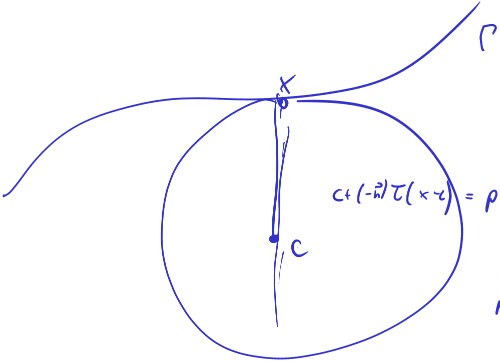
$\Delta u = 0$ everywhere but Γ

Idea:

1. Taylor-expand u about c
2. Evaluate Taylor expn at x
3. Done.

$$\sum_{|p|\le k} \frac{\partial_x^p u(x)|_{x=c}}{p!}(x-c)^p$$

$$= \int_\Gamma \underbrace{\sum_{|p|\le k} \frac{\partial_x^p \log(x-y)}{p!}(x-c)^p}\,\sigma(y)\,dS_y$$



$$c + (-\vec{n})\tau(x - c) = \rho(\tau)$$

$$\rho(0) = c$$

$$\rho(1) = x$$

$$\tau \in \mathbb{R}$$

$$\underline{\tau \mapsto S_0(\rho(\tau))}$$

Green's Formula

What if $\triangle v = 0$ and $u = G(|y - x|)$ in Green's second identity?

$$\int_{\Omega} \cancel{u \triangle v} - v \triangle u = \int_{\partial\Omega} u(\hat{n} \cdot \nabla v) - v(\hat{n} \cdot \nabla u) ds$$

Can you write that more briefly?

$$\boxed{v(x) = \int v \, \delta(x-y) \, dy = S(\partial_n v) - D(v)}$$

Green's Formula (Full Version)

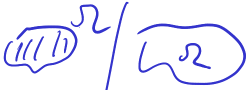
Ω bounded

Theorem (Green's Formula [Kress LIE 2nd ed. Thm 6.5])

If $\triangle u = 0$*, then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in \Omega, \\ \frac{u(x)}{2} & x \in \partial\Omega, \\ 0 & x \notin \Omega. \end{cases}$$

Green's Formula and Cauchy Data



Suppose I know '**Cauchy data**' $(u|_{\partial\Omega}, \hat{n}\cdot\nabla u|_{\partial\Omega})$ of u. What can I do?

Ev. Green's f., compute u.

What if Ω is an exterior domain?

?!

What if $u = 1$? Do you see any practical uses of this?

$$-D1 = \underbrace{\begin{cases} 1 & x \in \Omega \\ 1/2 & x \in \partial\Omega \\ 0 & \text{other.} \end{cases}}_{\text{indicator function}}$$

Mean Value Theorem

Theorem (Mean Value Theorem [Kress LIE 2nd ed. Thm 6.7]**)**

If $\Delta u = 0$, $u(x) = \overline{\int}_{B(x,r)} u(y)dy = \overline{\int}_{\partial B(x,r)} u(y)dy$ ✓

Define $\overline{\int}$?

$$|\Omega| = \int_\Omega 1\,dx \qquad \overline{\int}_\Omega f(x) = \frac{1}{|\Omega|}\int f(x)\,dx$$

Trace back to Green's Formula (say, in 2D):

$$u(x) = \int (\partial_n u)\Phi - \qquad \Phi(x) = \frac{1}{2\pi}\log(r)\cancelto{0}{\int_{\partial B}\partial_n u} - \frac{1}{2\pi r}\int_{\partial B} u.$$

Maximum Principle

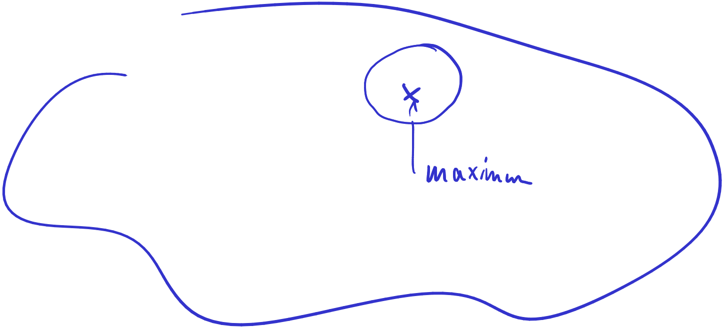
Theorem (Maximum Principle [Kress LIE 2nd ed. 6.9])

If $\triangle u = 0$ on compact set $\bar{\Omega}$:
u attains its maximum on the boundary.

Suppose it were to attain its maximum somewhere inside an open set. . .

contradicts mean value

What do our *constructed* harmonic functions (layer potentials) do there?


maximum

Green's Formula at Infinity: Statement

$\Omega \subseteq \mathbb{R}^n$ bounded, C^1, connected boundary, $\triangle u = 0$ in $\mathbb{R}^n \setminus \Omega$, u bounded

Theorem (Green's Formula in the exterior [Kress LIE 3rd ed. Thm 6.11])

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + \mathrm{PV} u_\infty = u(x)$$

for some constant u_∞. Only for $n = 2$,

$$u_\infty = \frac{1}{2\pi r} \int_{|y|=r} u(y) ds_y .$$

Realize the power of this statement:

Tells us about decay behavior of h.f. as $x \to \infty$.

Green's Formula at Infinity: Proof (1/4)

We will focus on $\mathbb{R}^3$. WLOG assume $0 \in \Omega$. Let $M = \|u\|_{L^\infty(\mathbb{R}^n \setminus \bar{\Omega})}$.
First, show $\|\nabla u\| \leq 6M / \|x\|$ for $x \geq R_0$.

Green's Formula at Infinity: Proof (2/4)

Let $x \in \mathbb{R}^3 \setminus \bar{\Omega}$. Let r be such that $\bar{\Omega} \subset B(x,r)$. Apply Green's formula on *bounded* domains to $B(x,r) \setminus \bar{\Omega}$:

$$(S_{\partial\Omega}(\partial_n u) - D_{\partial\Omega} u)(x) + (S_{\partial B(x,r)}(\partial_n u) - D_{\partial B(x,r)} u)(x) = u(x).$$

Show $S_{\partial B(x,r)}(\partial_n u) \to 0$ as $r \to \infty$:

Green's Formula at Infinity: Proof (3/4)

It remains to bound the term

$$D_{\partial B(x,r)}u)(x) = \frac{4\pi}{r^2}\int_{\partial B(x,r)} u(y)dS_y.$$

Can we transplant that ball to the origin in some sense?

Green's Formula at Infinity: Proof (4/4)

Observe

$$\left| \frac{4\pi}{r^2} \int_{\partial B(0,r)} u(y) dS_y \right| \leq 4\pi M.$$

Consider the sequence

$$\mu_n := \frac{4\pi}{r_n^2} \int_{\partial B(0,r_n)} u(y) dS_y.$$

Because of its boundedness and sequential compactness of the bounding interval, out of a sequence of radii r_n, we can pick a subsequence so that $(\mu_{n(k)})$ converges. Call the limit u_∞.

Green's Formula at Infinity: Impact

Can we use this to bound u as $x \to \infty$?
Consider the behavior of the kernel as $r \to \infty$. Focus on 3D for simplicity.
(But 2D holds also.)

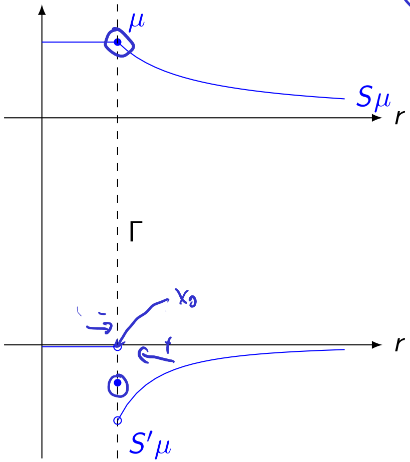
$$u(x) = u_\infty - S(\partial_n u) + D u = O\left(\frac{1}{r}\right)$$

How about u's derivatives?

$$\nabla u(x) = O\left(\frac{1}{r^{n-1}}\right) = \begin{cases} \frac{1}{r} & 2D \\ \frac{1}{r^2} & 3D \end{cases}$$

Outline

Jump relations:



Jump Relations: Mathematical Statement

Let $[X] = X_+ - X_-$. (Normal points towards "+"="exterior".)

"jump"

Theorem (Jump Relations [Kress LIE 2nd ed. Thm. 6.14, 6.17,6.18])

$(x_0 \in \partial\Omega)$

$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I\right)(\sigma)(x_0) \quad \Rightarrow \quad [S'\sigma] = -\sigma$$

$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I\right)(\sigma)(x_0) \quad \Rightarrow \quad [D\sigma] = \sigma$$

$$[D'\sigma] = 0$$

Truth in advertising: Assumptions on Γ?

$$\boxed{\partial\Omega \in C^2}$$

Jump Relations: Proof Sketch for SLP

Sketch the proof for the single layer.

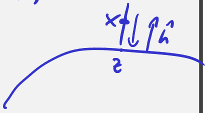
(Follow proof for weakly singular ⇒ bounded)

Jump Relations: Proof Sketch for DLP

Sketch proof for the double layer.

Suppose x is tgt point. (near Γ)

$$x = z + h\,\hat{n}(z) \qquad (z\in\Gamma)$$



$$D\sigma(x) = \underline{D(\sigma(z))} + D\sigma(x) + \underline{D(\sigma(z))}$$

$$= \sigma(z)\,D1 + \underbrace{D[\sigma - \sigma(z)](x)}_{\to 0}$$

As $h\to 0$ (as $x\to z$), $\sigma(x)\to\sigma(z)$,

Outline

Outline

Boundary Value Problems: Overview

$\Delta u = 0 \quad \Delta X = 0$

	Dirichlet	**Neumann**
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ ⊕ unique	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ ⓞ may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\|x\| \to \infty$ ⊕ unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\|x\| \to \infty$ ⊕ unique

with $g \in C(\partial\Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

$f(x) = O(g(x))$

$\lim \frac{f(x)}{g(x)} \leq C$

$f(x) = o(g(x))$

$\lim \frac{f(x)}{g(x)} = 0$

Uniqueness Proofs

Dirichlet uniqueness: why?

Neumann uniqueness: why?