April 23, 2024

Announcements

- HW4 due
- Projects

Goals

- $N(\frac{1}{2}+D) = \operatorname{span}\{1\}$
- ext. Dirichlet
- Helmholtz

Review

- BVP uniqueness
  - ↳ IE "uniqueness"

Uniqueness of Integral Equation Solutions

Fredholm

Theorem (Nullspaces [Kress LIE 3rd ed. Thm 6.21])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, where $\int \psi \neq 0$.

Fredholm alt.:

$(I - A)(x) = (N(I - A^*))^\perp$

has to be true for all harmonic f.

$\int \partial_n u = 0$

int. Neumann $\to$ $(I/2 + S')(x) \perp N(\frac{I}{2} + D) = \text{span}\{1\}$

ext. Dirichlet. $\to$ $(I/2 + D)(x) \perp N(\frac{I}{2} + S') = \text{span}\{\psi\}$

???

IE Uniqueness: Proofs (1/3)

Show $N(I/2 - D) = \{0\}$.

Let $\underbrace{(\frac{I}{2} - D)}_{\lim u}\psi = 0$. $u(x) := D\psi$.

$\cancel{\psi = [D\psi] = u^+ - \underbrace{u^-}_{0} = u^+}$.

$u = 0$ on int. (vol.)

$(\partial_n u)_- = 0$

$[\partial_n u] = [D'\psi] = 0 \Rightarrow (\partial_n u)^+ = 0$

$u = 0$ on ext. (vol.)

$\Rightarrow \psi = 0$.

IE Uniqueness: Proofs (2/3)

Show $N(I/2 - S') = \{0\}$.

IE Uniqueness: Proofs (3/3)

Show $N(I/2 + D) = \text{span}\{1\}$.

„$\subseteq$“ Let φ s.t. $\underbrace{\frac{\varphi}{2} + D = 0}_{\text{lin} + u}$. $\quad u(x) = D\varphi(x) \Rightarrow u^+ = 0.$

$\Rightarrow u \equiv 0$ in ext. (vol.)

$(\partial_n u)^+ = 0 \Rightarrow (\partial_n u)^- = 0 \Rightarrow u \equiv$ const. in int. vol.

$\varphi = [D\varphi] = u^+ - u^- = 0 -$ const. $\Rightarrow \varphi \in \text{span}\{1\}$

„$\supseteq$“: ✓ because of D1

What conditions on the RHS do we get for int. Neumann and ext. Dirichlet?

... (see above)

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ do not know ψ. Assume $0 \in \Omega$. $\mathbb{R}^3$

$K(x,y) = \partial_{n_y} G(x,y) \quad \rightsquigarrow \quad \tilde{K}(x,y) = \partial_n G(x,y) + \frac{1}{|x|^{n-2}}$

$u(x) := \int \tilde{K}(x,y)\sigma(y)\, dS_y = D\sigma + \frac{1}{|x|^{n-2}} \int \sigma(y)\, dS_y$

- Solves PDE, stays compact, no add'l jumps, ext. limit OK.
- ext. Dirichlet uniqueness $\Rightarrow u \equiv 0$ in ext. vol.

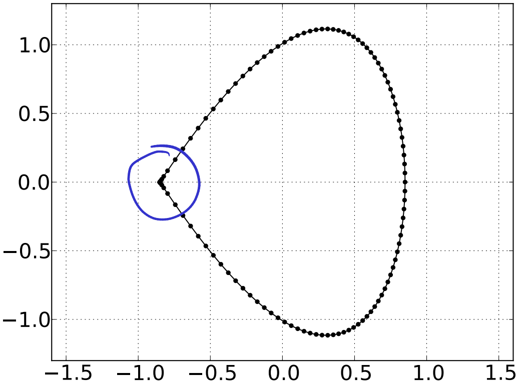
$0 = |x|^{n-2} u(x) \xrightarrow[r]{} \int \sigma + O(\frac{1}{r}) \to \int \sigma$ (as $x \to \infty$)

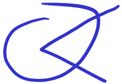
if $D\sigma = O(\frac{1}{r^2})$ in 3D

$\frac{\sigma}{2} + D\sigma = 0 \Rightarrow \sigma \in N(\frac{1}{2} + D) = \operatorname{span}\{1\} \Rightarrow \int \sigma = 0$ σ const

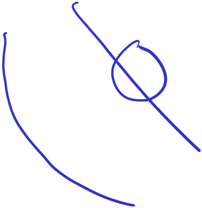
$\Rightarrow \sigma = 0.$

Domains with Corners

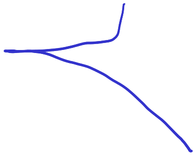




$D1 = \{ -\frac{1}{2}$ u o ∂Ω



What's the problem?



Domains with Corners (II)

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x,y)\phi(y) ds_y \pm \frac{1}{2} \frac{\langle \text{opening angle on} \pm \text{side} \rangle}{\pi} \phi$$

Name some problems.

$u(x) := D\sigma(x$

$\lim_{x \to \partial -} u(x) = D\sigma \begin{cases} -\sigma/2 \\ -\sigma \cdot ?? \end{cases}$ $\rightsquigarrow$ not second kind??

Workarounds?

- It bounded + compact
- C^2

Numerically: Needs consideration, can drive up cost through refinement.

Outline

Where does Helmholtz come from?

Derive the Helmholtz equation from the wave equation $\partial_t^2 U = c^2 \triangle U$,. Q: What is c?

$c =$ prop speed $\quad [c] = \frac{m}{s}$

$$U(x,t) = u(x)\, e^{-i\omega t}$$

$$u(x)\, \partial_t \left(e^{-i\omega t}\right) = c^2 e^{-i\omega t} \Delta u(x)$$

$$u(x) \left(-i\omega\right)^2 \cancel{e^{-i\omega t}} = c^2 \cancel{e^{-i\omega t}} \Delta u(x)$$

$$-\omega^2 u(x) = c^2 \Delta u(x)$$

$$\Delta u + \underbrace{\left(\frac{\omega^2}{c^2}\right)}_{k} u(x) = 0$$

"wave number"

$$[k] = \frac{\frac{1}{s}}{\frac{m}{s}} = \frac{1}{m}$$

Helmholtz vs. Yukawa

"bad"

"good"

Helmholtz Equation

- $\triangle u + k^2 u(x) = 0$
- Indefinite operator
- Oscillatory solution
- Difficult to solve, especially for large k

Yukawa Equation

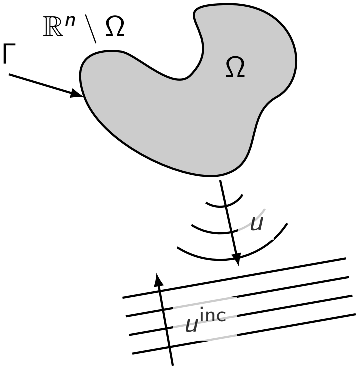
- $-\triangle u + k^2 u(x) = 0$
- Positive definite operator
- Smooth solutions
- 'Screened Coulomb' interaction
- Generally quite simple to solve

$-\Delta$ is pos. def.

$$\int \Delta u \, \varphi$$

$$= - \int \nabla u \cdot \nabla \varphi \overset{u=\varphi}{>} 0$$

The prototypical Helmholtz BVP: A Scattering Problem



Ansatz:

$$u^{\text{tot}} = u + u^{\text{inc}}$$

Solve for scattered field u.

decays

Helmholtz: Some Physics

Physical quantities:

- Velocity potential: $U(x,t) = u(x)e^{-i\omega t}$
  (fix phase by e.g. taking real part)
- Velocity: $v = (1/\rho_0)\nabla U$
- Pressure: $p = -\partial_t U = i\omega u e^{-i\omega t}$
  - Equation of state: $p = f(\rho)$

"sound-hard"

"sound-soft"

What's ρ_0?

"base" density in lin. of Euler

What happens to a pressure BC as $\omega \to 0$?

"disappears"

Helmholtz: Boundary Conditions

Interfaces between media: What's continuous?

- Sound-soft: Scatterer "gives"
  - Pressure remains constant in time
  - $u = f \to$ Dirichlet
- Sound-hard: Scatterer "does not give"
  - Pressure varies, same on both sides of interface
  - $\hat{n} \cdot \nabla u = 0 \to$ Neumann
- Impedance: Some pressure translates into motion
  - Scatterer "resists"
  - $\hat{n} \cdot \nabla u + ik\lambda u = 0 \to$ Robin ($\lambda > 0$)
- Sommerfeld radiation condition: allow only outgoing waves (n-dim)

$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - ik \right) u(x) \to 0 \qquad (r \to \infty)$$

Many interesting BCs $\to$ many IEs! :)