CS 598: Provably Efficient Algorithms for Numerical and Combinatorial Problems

Part 3: Parallelism in Algorithms

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Circuits and PRAM

- Circuits were the first parallel algorithms

- The *PRAM* model tries to stay consistent with this view

Inner Product in the PRAM Model

- Inner product with n processors

- Inner product with $n/\log_2(n)$ processors

Basic Linear Algebra Subroutines (BLAS) in the PRAM Model

- Vector scaling (BLAS 1)

- Matrix-vector multiplication and outer product (BLAS 2)

Work-Depth Model

- The work-depth (or work-time) model keeps track only of total work and algorithm depth/time

- Its possible to schedule a work-optimal PRAM algorithm so that it uses an asymptotically optimal number of processors

Numerical Linear Algebra in PRAM

- Standard algorithms for triangular solve and matrix factorizations have polynomial depth

- Polylogarithmic depth algorithms exist for solving linear systems

Recursive Matrix Factorization Depth

- Recursive Cholesky $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T$ has polynomial depth

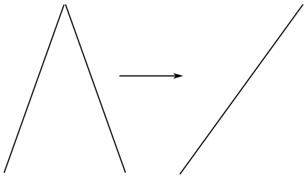
- Recursive triangular inversion $\boldsymbol{S} = \boldsymbol{L}^{-1}$ has logarithmic depth

Sorting and Parallel Sorting

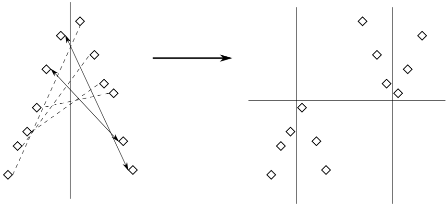
- Parallel sorting within a single shared-memory

- Most sorting algorithms can be classified as *merge-based* or *distribution-based*

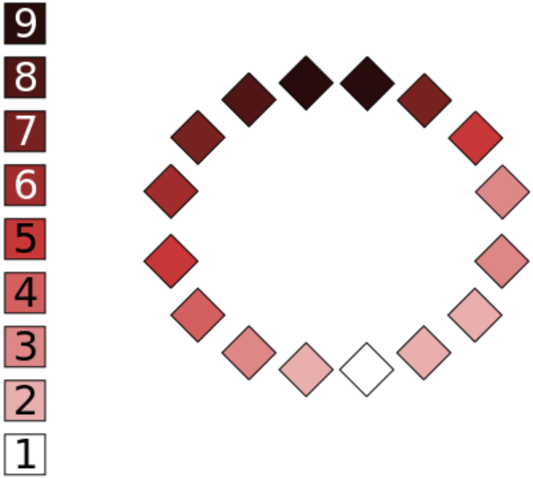
Bitonic Sort



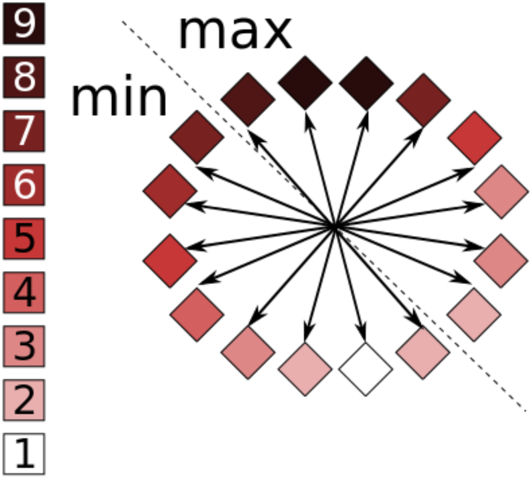
Bitonic Merge



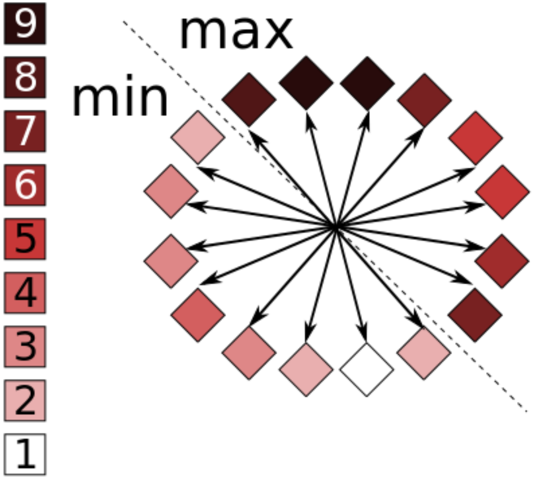
Bitonic sequence as a circle



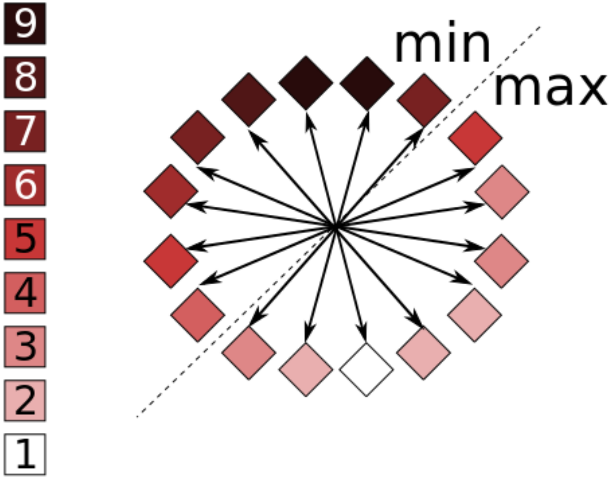
Matching opposite pairs in the circle



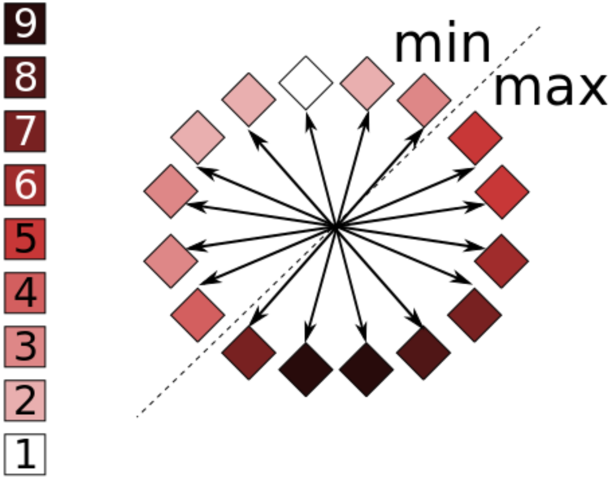
Swapping opposite pairs in the circle



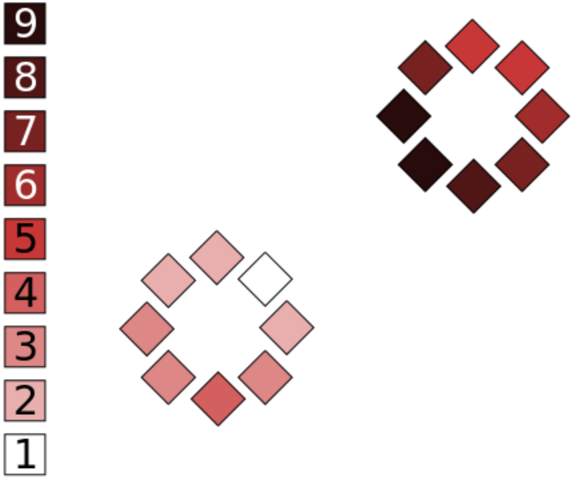
Collecting the min/max into different subsequences



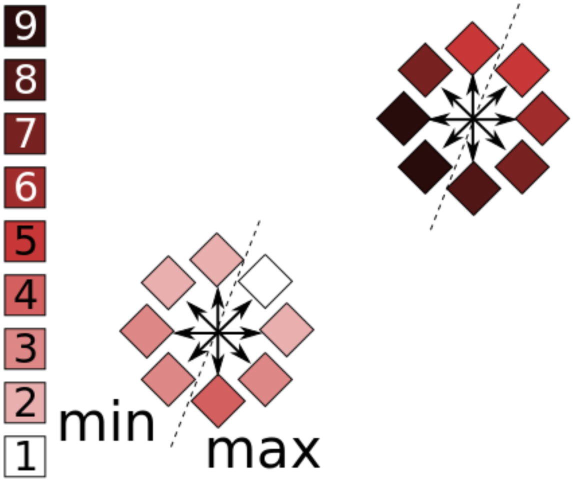
Any partition subdivides smaller/greater halves



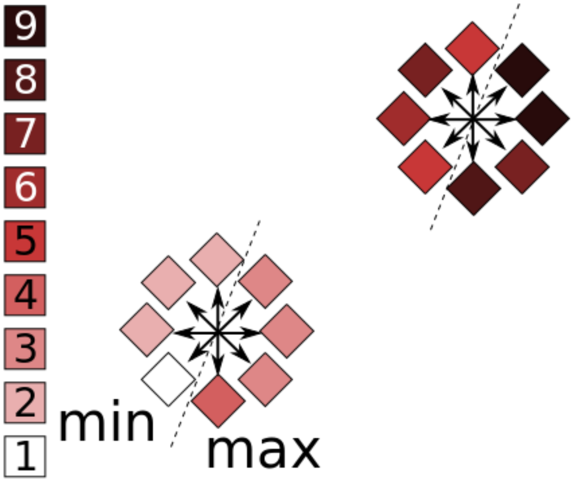
Arranging the two halves into new circles



Swapping opposites again



Continuing with bitonic merge recursively



Bitonic merge

- A *bitonic sequence* is any cyclic shift of the sequence $\{i_0 \leq \cdots \leq i_k \geq \cdots i_{n-1}\}$

- There exists $l \leq k$, such that the largest $n/2$ elements of (unshifted bitonic sequence) S are the subsequence $\{i_l, \ldots, i_{l+n/2-1}\}$

BFS with Sparse Linear Algebra

- For undirect graph $G = (V, E)$ Breadth First Search (BFS) takes as input a source vertex s and outputs an assignment of vertices to frontiers

- With adjacency matrix $\boldsymbol{A}$ of G, can compute BFS via matrix-vector products

Sparse Linear Algebra in PRAM

- Sparse-matrix-vector product (SpMV) with m nonzeros (edges) in matrix

- Sparse-matrix-sparse-vector product (SpMSpV) with k nonzeros (frontier vertices) in vector

BFS on a PRAM

- Each BFS iteration requires an SpMSpV with an output filter
  $\boldsymbol{f}^{(i+1)} = \boldsymbol{u}^{(i)} \odot (\boldsymbol{A}\boldsymbol{f}^{(i)})$

- Different choices of BFS algorithm yield different work/depth

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Connectivity in Graphs

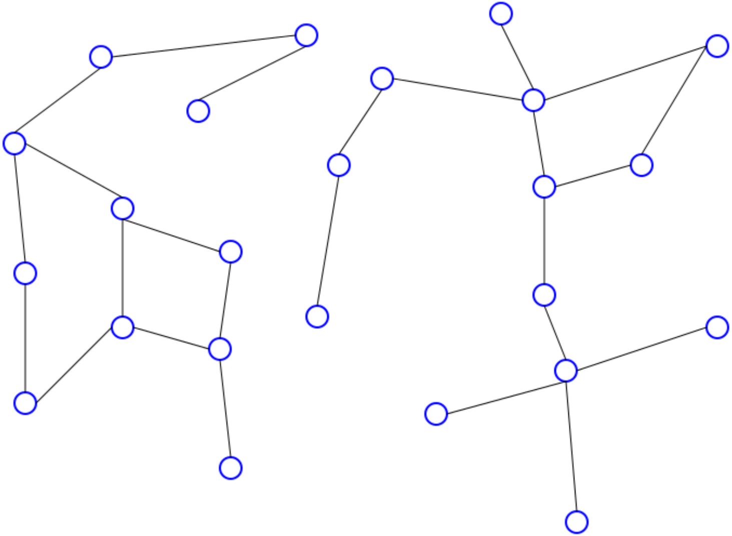
- Connectivity seeks to label vertices with a unique label for each connected component
- Shiloach and Vishkin (1980) CRCW PRAM algorithm for connectivity

Shiloach-Vishkin Connectivity Algorithm

Let each node i store 'parent' $p(i)$ and perform below steps until convergence

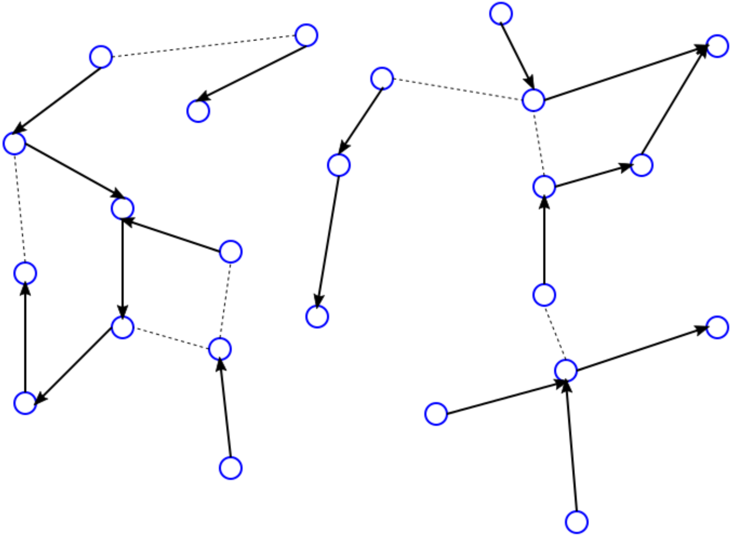
- conditional star hooking
- unconditional star hooking
- Shortcutting (pointer chasing)

A graph with two connected components



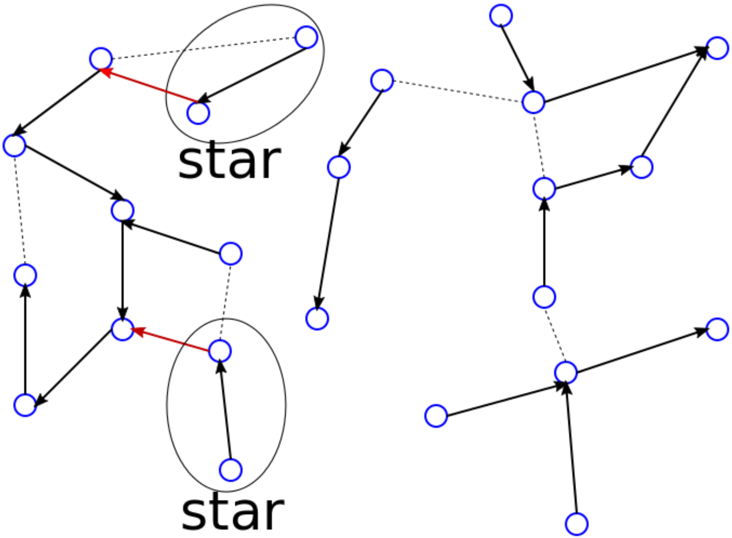
First iteration

1. conditional star hooking



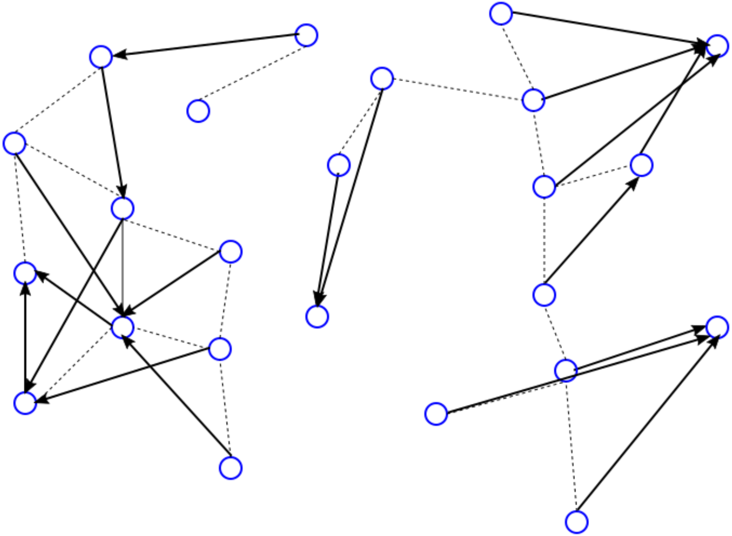
First iteration

2. unconditional star hooking



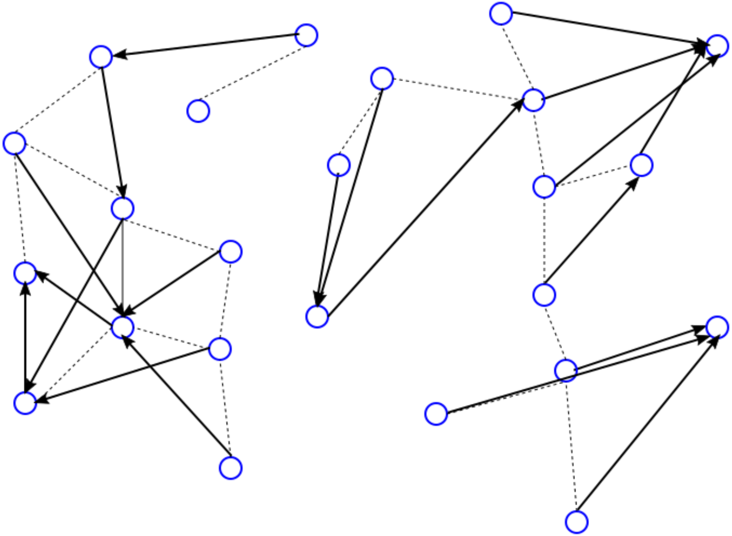
First iteration

3. shortcutting



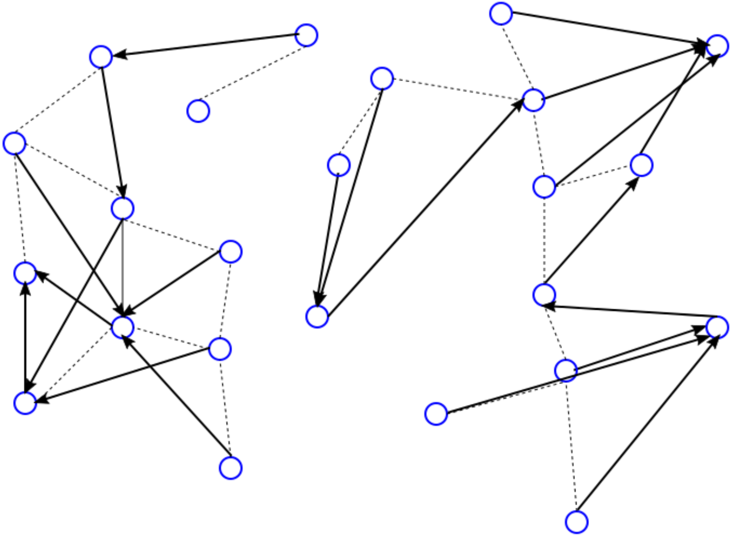
Second iteration

1. conditional star hooking



Second iteration

2. unconditional star hooking



Analysis of parallel tree connectivity

Algorithm converges after $O(\log(n))$ iterations

- Sum of tree heights (starts at n) decreases by a factor of at least $3/2$ every iteration

- Requires $O(n + m)$ work per iteration

Shortest Paths

- Given a positive weight function $w : E \to \mathbb{R}^+$, compute shortest distances from a source vertex s to all other vertices
- Bellman-Ford can be expressed as matix-vector products on the tropical (min–plus) semiring, using SpMV/SpMSpV

All-Pairs Shortest-Paths

- Given a positive weight function $w : E \to \mathbb{R}^+$, compute shortest distances matrix $\boldsymbol{D}$ containing minimum distances between all pairs of vertices

- Floyd-Warshall algorithm computes achieves $O(n^3)$ work

Floyd Warshall Algorithm

- $\boldsymbol{D}^{(i)}$ contains the distances of all shortest paths $S^{(i)}$ with at most i edges going through some subset of vertices $\{1, \ldots, i-1\}$

- A recursive alternative to Floyd-Warshall is given by Gauss-Jordan elimination (Kleene's APSP algorithm)

Parallel All-Pairs Shortest-Paths

- Path doubling can be used to obtain polylogarithmic depth
- Tiskin (2001) proposed an improvement to achieve $O(n^3)$ cost

Parallel (Approximate) Matrix Inversion

- Gauss-Jordan can be used to invert matrix, recursive Cholesky is similar
- Can theoretically invert with polylogarithmic depth via Newton's method