1. Consider the two-point boundary value problem on $\Omega = [0, 1]$,
\[-\nu \frac{d^2 \tilde{u}}{dx^2} + \beta \tilde{u} = f, \quad \tilde{u}(0) = \tilde{u}(1) = 0,
\]
for constant parameters $\nu > 0$ and $\beta > 0$. The problem is discretized using linear finite elements with $u \in X_0^N \subset X^N$, where $X^N = \{\phi_0, \ldots, \phi_N\}$ is the set of piecewise linear Lagrange basis functions on nodal points $0 = x_0, x_1, \ldots, x_N = 1$. The FEM Galerkin discretization leads to a system for the unknowns of the form
\[
\nu A u + \beta B u = b.
\]
Here, the respective stiffness and mass matrices, $A = R \bar{A} R^T$ and $B = R \bar{B} R^T$ are computed as the restriction of their extended counterparts, $\bar{A}$ and $\bar{B}$, where $R$ is the $(N-1) \times (N+1)$ restriction matrix.

Which of the following is true of the entries $b_{ij}$ of the extended mass matrix, $\bar{B}$?

(a) $\sum_{j=0}^{N} b_{ij} = 0.$

(b) $\sum_{j=0}^{N} b_{ij} = 1.$

(c) $\sum_{i=0}^{N} b_{ij} = 2.$

(d) $\sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} = 0.$

(e) $\sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} = 1.$
Solution

Let \( u(x) = 1 \) and \( v(x) = 1 \):

\[
\sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} = \sum_{i=0}^{N} \sum_{j=0}^{N} v_i b_{ij} u_j \\
= \sum_{i=0}^{N} \sum_{j=0}^{N} v_i \int_{\Omega} \phi_i \phi_j \, dx \, u_j \\
= \sum_{i=0}^{N} \sum_{j=0}^{N} \int_{\Omega} v_i \phi_i \phi_j \, dx \\
= \int_{\Omega} v u \, dx \\
= \int_{\Omega} 1 \cdot 1 \, dx \\
= 1
\]

Thus, the sum of all the entries in the \( \bar{B} \) matrix is the integral of 1 over the whole domain.
2. Consider the two-point boundary value problem on \( \Omega = [0, 1] \),

\[
- \frac{d^2 \tilde{u}}{dx^2} = 1, \quad \tilde{u}(0) = \tilde{u}(1) = 0.
\]

The analytical solution is \( \tilde{u}(x) = \frac{1}{2}(x - x^2) \), for which slope at \( x = 0 \) is \( \tilde{u}'(0) = \frac{1}{2} \).

The problem is discretized using collocation with a single nodal point, \( x_1 = 1/2 \), and basis function \( \phi_1(x) = \sin \pi x \). The numerical solution thus has the form \( u(x) = \hat{u}_1 \sin \pi x \).

Which of the following is the best estimate of the derivative of the numerical solution at \( x = 0 \), \( u'(0) \)?

(a) \( \frac{2}{3} \)
(b) \( \frac{1}{2} \)
(c) \( \frac{1}{3} \)
(d) \( \frac{1}{4} \)
(e) \( \frac{1}{5} \)

Solution

Start with substitution of the solution into the ODE:

\[
- \frac{d^2 \tilde{u}}{dx^2} = 1
\]

\[
-(-\pi^2 \hat{u}_1 \sin(\pi x)) = 1
\]

\[
\pi^2 \hat{u}_1 \sin(\pi x) = 1
\]

\[
\pi^2 \hat{u}_1 \sin(\pi(1/2)) = 1
\]

\[
\pi^2 \hat{u}_1 = 1
\]

\[
u_1 = \frac{1}{\pi^2}
\]

Then evaluate the derivative at \( x = 0 \):

\[
u'(0) = \hat{u} \pi \cos(\pi 0) = \frac{1}{\pi}
\]

Therefore, (c) is the correct answer.
3. The two-point boundary value problem on $\Omega = [0, 1]$,

$$\frac{-d^2 \ddot{u}}{dx^2} = f, \quad \ddot{u}(0) = \ddot{u}(1) = 0.$$ 

is discretized with a Galerkin method using $N$th-order Lagrange polynomial interpolants $\phi(x)$ on $N+1$ Gauss-Lobatto-Legendre quadrature points that are mapped to $\Omega = [0, 1]$. The extended stiffness matrix has entries

$$\bar{A}_{ij} = \int_0^1 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} \, dx \quad \{ \begin{array}{l} i = 0, \ldots, N \\ j = 0, \ldots, N \end{array}.$$ 

Which of the following best estimates the minimum eigenvalue of $\bar{A}$?

(a) -1 
(b) 0 
(c) 1 
(d) $\pi$ 
(e) $\pi^2$

Solution

The eigenvalue problem is:

$$\bar{A} \mathbf{v} = \lambda \mathbf{v}$$ 
$$\mathbf{v}^T \bar{A} \mathbf{v} = \mathbf{v}^T \lambda \mathbf{v}$$ 

$$\sum_{i=0}^{N} \sum_{j=0}^{N} v_i a_{ij} v_j = \sum_{i=0}^{N} \lambda v_i^2$$ 

$$\sum_{i=0}^{N} \sum_{j=0}^{N} \int_{\Omega} \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} v_j \, dx = \sum_{i=0}^{N} \lambda v_i^2$$ 

$$\sum_{i=0}^{N} \sum_{j=0}^{N} \int_{\Omega} \frac{dv_i}{dx} \frac{dv_j}{dx} \phi_i \phi_j \, dx = \lambda \sum_{i=0}^{N} v_i^2$$ 

$$\int_{\Omega} \frac{dv}{dx} \frac{dv}{dx} \, dx = \lambda \sum_{i=0}^{N} v_i^2$$ 

Therefore, if $v$ is a vector with the same value for every element, it would have a corresponding eigenvalue of $\lambda = 0$. Furthermore, the integral must always evaluate to a non-negative number and the sum on the right hand side is also non-negative so $\lambda$ cannot be negative. The correct answer is (b).
4. A 5th-order Lagrange polynomial is used to interpolate \( f(x) = e^{\cos x} \) with nodal points \( x_j \in [0, L] \) and takes the form

\[
p(x) = \sum_{j=0}^{5} f_j \phi_j(x).
\]

As \( L \to 0 \), we can expect convergence of the form

\[
\max_{x \in [0, L]} |p(x) - f(x)| = O(L^k).
\]

What is \( k \)?

Solution

For an \( n \)th order polynomial, we can start with the error formula and bound the error:

\[
f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n)
\]

\[
\leq \frac{\max_{x \in [0, L]} f^{(n+1)}}{(n+1)!} L^{n+1}
\]

Therefore, \( k = n + 1 = 6 \).

5. A 1st-order Lagrange polynomial, \( p(x) \), is used to interpolate \( f(x) = \cos e^x \) with nodal points \( x_j \in [0, L] \). As \( L \to 0 \), we can expect convergence of the form

\[
\max_{x \in [0, L]} |p(x) - f(x)| = O(L^k).
\]

What is \( k \)?

Solution

From the previous problem, \( k = n + 1 = 2 \).
6. Ignoring round-off effects, what is the maximum error to be expected if a 4th-order polynomial is used to approximate \( \cos x \) on the interval \([0, 1]\) for any distribution of distinct nodal points \( x_j \in [0, 1] \)?

Specifically, which of the following is closest to this error bound?

(a) 0.1  
(b) 0.01  
(c) 0.001  
(d) 0.0001

**Solution**

Starting again with the error formula and substituting \( n = 4 \):

\[
 f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)(x - x_1) \ldots (x - x_n) 
\]

\[
 \leq \max_{x \in [0, L]} f^{(n+1)} L^{n+1} 
\]

\[
 \leq \frac{\max_{x \in [0, L]} f^{((4)+1)}}{(4+1)!} (1)^{(4)+1} 
\]

\[
 \leq \frac{1}{120} 
\]

So the correct answer is (b).

7. It is desired to have a one-sided approximation to the second derivative of \( u(x) \) at \( x \) involving \( u_{j-2} := u(x_{j-2}), u_{j-1} := u(x_{j-1}), \) and \( u_j := u(x_j) \).

Assuming \( x_0 = -1, x_1 = 0, x_2 = 1 \), we can approximate this second derivative as

\[
 \left. \frac{d^2 u}{dx^2} \right|_{x=1} = au_0 + bu_1 + cu_2. 
\]

**What are \( a, b, \) and \( c \)?**

**Solution**

We can find the Taylor expansion of the the solution at the various points about \( x = x_2 \).

\[
 u_2 = u_2 
\]

\[
 u_1 = u_2 - u_2' + \frac{u_2''}{2} + \text{h.o.t.} 
\]

\[
 u_0 = u_2 - 2u_2' + 4\frac{u_2''}{2} + \text{h.o.t.} 
\]
The linear combination can be expressed as:

\[ au_0 + bu_1 + cu_2 = a(u_2 - 2u'_2 + \frac{u''_2}{2} + \text{h.o.t.}) + b(u_2 - u'_2 + \frac{u''_2}{2} + \text{h.o.t.}) + cu_2 \]
\[ = (a + b + c)u_2 + (-2a - b)u'_2 + (2a + b/2)u''_2 + \text{h.o.t.} \]

For this problem, we want the coefficient in front of \( u_2 \) and \( u'_2 \) to be 0, and the coefficient in front of \( u''_2 \) to be 1. Therefore: \( a + b + c = 0 \), \( 2a + b = 0 \), \( 2a + b/2 = 1 \). When the system of linear equations is solved, we get:

\[ a = 1, \]
\[ b = -2, \]
\[ c = 1. \]

8. Consider the initial value problem

\[ \frac{du}{dt} = 4u, \quad u(0) = 1. \]

Euler forward timestepping is used to advance this solution with \( u^0 = u(0) = 1 \) and timestep size \( \Delta t = 0.1 \).

**What is \( u^1 \)?**

**Solution**

We first start by discretizing the time-derivative term:

\[ \frac{u^n - u^{n-1}}{\Delta t} = 4u^{n-1} \]
\[ u^n = u^{n-1} + 4\Delta tu^{n-1} \]
\[ u^n = 1.4u^{n-1} \]

Because \( u^0 = 1, u^1 = 1.4u^0 = 1.4 \).

*Note if you’ve interpreted \( u^1 \) to be \( u(1) \), you also received credit.*
9. We wish to solve the spring-mass system,

\[ m \frac{d^2 y}{dt^2} = -ky, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 0, \]

where \( m \) is the mass, \( k \) is the stiffness, and \( y \) is the displacement as a function of time, \( t \).

Let \( y^j \) represent numerically-computed position of the mass at time \( t^j = j \cdot \Delta t \) and \( \dot{y}^j \) represent the corresponding velocity.

Suppose we advance the system using Euler forward applied to an equivalent first-order system for \([y \ \dot{y}]^T\), as we did in HW 1. Let \( k=1, m=1, \) and \( \Delta t = 0.1 \).

**What is \( y^1 \)?**

**What is \( \dot{y}^1 \)?**

**Solution**

Let \( q_0 = y \) and \( q_1 = \dot{y} \):

\[
\begin{align*}
\frac{dq}{dt} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} q \\
\frac{dq}{dt} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} q
\end{align*}
\]

We again discretize the time-derivative term:

\[
\begin{align*}
\frac{q^n - q^{n-1}}{\Delta t} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} q^{n-1} \\
q^n &= q^{n-1} + \Delta t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} q^{n-1} \\
q^n &= (I + \Delta t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) q^{n-1} \\
q^n &= \begin{bmatrix} 1 & 0.1 \\ -0.1 & 1 \end{bmatrix} q^{n-1}
\end{align*}
\]

Therefore, \( q^1 = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix} \). Then \( y^1 = 1 \) and \( \dot{y}^1 = -0.1 \).

*Note if you’ve interpreted \( y^1 \) to be \( y(1) \) and \( \dot{y}^1 \) to be \( \dot{y}(1) \), you have received credit.*
10. Consider the boundary value problem
\[- \frac{d^2 \tilde{u}}{dx^2} = 1, \quad \Omega = [0, 2]\]
\[- \left. \frac{d\tilde{u}}{dx} \right|_{x=0} = 2, \quad \tilde{u}(2) = 0.\]

This problem is solved with linear finite elements with grid spacing \( h = 1 \) and nodal points,
\[x_0 = 0, \quad x_1 = 1, \quad x_2 = 2,\]
leading to the linear system for the vector of unknown basis coefficients, \( u \),
\[Au = b, \quad u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad b = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}.\]

- What is \( b_0 \)?
- What is \( b_1 \)?
- What is \( u_0 \)?
- What is \( u_1 \)?

**Solution**

First, we know what \( \bar{A} \) and \( \bar{B} \) are:
\[
\bar{A} = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]
\[
\bar{B} = \frac{1}{6} \begin{bmatrix}
2 & 1 & 0 \\
1 & 4 & 1 \\
0 & 1 & 2
\end{bmatrix}
\]

We also need a restriction matrix that truncates a vector on the bottom:
\[R = \begin{bmatrix} I & 0 \end{bmatrix}\]

Where \( R \) is a 2 by 3 matrix.

We can re-write the problem statement in linear algebra form:
\[R\bar{A}R^T u = R (\bar{B}^T + \bar{y})\]
Where $\mathbf{f} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{g} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ — the negative flux.

Therefore,

$$b = -R (\bar{A} \mathbf{u}_b + \mathbf{g})$$

$$= \begin{pmatrix} 5/2 \\ 1 \end{pmatrix}$$

The system that needs to be solve is:

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{u} = \begin{pmatrix} 5/2 \\ 1 \end{pmatrix}$$

The solution is

$$\mathbf{u} = \begin{pmatrix} 6 \\ 7/2 \end{pmatrix}$$