TAM470 Fall 2018 Homework Set 2. Due October 1, 5 PM.

1. **30 Points.** Consider the one-dimensional heat equation on \( \Omega = [-1, 1] \),

\[
- \frac{d}{dx} \left( k \frac{du}{dx} \right) = 0, \quad \text{for} \quad u(-1) = 1, \quad u(1) = 0,
\]

with \( k(x) = 10 \) for \( x < 0 \), \( k(x) = 1 \) for \( x \geq 0 \).

a. Write down the weighted residual formulation for this problem.

b. Write a matlab code to find a solution to this problem using finite elements with a linear basis. In particular, calculate \(-k \frac{du}{dx}(x = 1)\) (physically, this would be the flux) good to at least 1%.

What is the flux at \( x = -1/2 \)?

c. Plot your flux estimate as a function of the number of elements. Plot your solution, \( u(x) \), for the finest mesh you considered.

**Solution**

1a.

\[
- \int_{\Omega} v \frac{d}{dx} \left( k \frac{du}{dx} \right) dx = 0 \quad (1)
\]

\[
\int_{\Omega} k \frac{dv}{dx} \frac{du}{dx} dx - kv \frac{du}{dx} \bigg|_{-1}^{1} = 0 \quad (2)
\]

\[
\int_{\Omega} k \frac{dv}{dx} \frac{du}{dx} dx = 0 \quad (3)
\]

\[
\int_{-1}^{0} 10 \frac{dv}{dx} \frac{du}{dx} dx + \int_{0}^{1} \frac{dv}{dx} \frac{du}{dx} dx = 0 \quad (4)
\]

\[
\int_{-1}^{0} 10 \frac{dv}{dx} \frac{du}{dx} dx + \int_{0}^{1} \frac{dv}{dx} \frac{du}{dx} dx = 0 \quad (5)
\]

Find \( u \in X^N \) s.t.

\[
\int_{-1}^{0} 10 \frac{dv}{dx} \frac{du}{dx} dx + \int_{0}^{1} \frac{dv}{dx} \frac{du}{dx} dx = 0 \quad (6)
\]

, where \( v \in X^N_0 \)

1b.

In order to solve this problem in Matlab the line:

\[
Ab = D^T*B*D;
\]

must be changed to:

\[
k=@(x) 5.5-4.5*sign(x); Ab = D^T*B*sparse(diag(k(zq)))*D;
\]

This change is the implementation of the variable diffusivity. In order to account for the inhomogeneous Dirichlet BC,
\[ b = R \cdot J_h' \cdot B \cdot J_h \cdot f; \]
must be changed to
\[ b = - R \cdot A_b \cdot u_b; \]
The flux at the end can be obtained by applying the derivative matrix to the solution and scaling by \( k \). Although the derivatives are evaluated at the mid-points, there is no ambiguity for the derivative value at the end since there is only 1 midpoint value that is the closest. The correct value is: \(-k \frac{du}{dx} = 0.909\)

1c.

Plot of flux vs. \( N \).

Plot of highest resolution solution:

Here, use Gauss-Lobatto-Legendre (GLL) quadrature with \((N + 1)\) points on the interval \([-L, L]\). Choose \(N\) and \(L\) sufficiently large to obtain an error < 1%. Discuss how you chose \(L\). For your given \(L\), plot error vs. \(N\) on a semi-log plot.

Note, you may use the `zwgl11.m` script to get the GLL quadrature points and weights.

**Solution**

Since the integrand has an exponential factor, if \(L\) becomes large enough, the ratio of the integrand value at \(x = 0\) and \(x = L\) becomes smaller than machine precision. Therefore, we just need to find \(L\) s.t. \(e^{-L^2} = 10^{-16}\). From this relation, \(L \approx 6\):
At $x = L = 6$, we know that the integrand must be less than machine precision since $|\cos(x)| \leq 1 \forall x$.

Now that we have $L$, we can get the error behavior vs. $N$: 
From this plot, it is clear that for $L = 6$, the definite integral has error of less than 1% for $N \gtrsim 35$.

3. **30 Points.** Consider the one-dimensional Helmholtz equation in cylindrical coordinates,

$$\frac{d}{dr} \left( r \frac{du}{dr} \right) + u = 0, \quad u(r = 0) \text{ bounded, } u(r = 1) = 1.$$

a. Write down the weighted residual formulation for this problem.

b. Write a matlab code to find a solution to this problem using finite elements with a linear basis. In particular, calculate the flux $-\frac{du}{dr}(r = 1)$ good to at least 1%.

c. Plot your flux estimate as a function of the number of elements. Plot your solution, $u(x)$, for the finest mesh you considered.
Solution

a.

\[-\frac{d}{dr}\left(r\frac{du}{dr}\right) + ur = 0\]  \hspace{1cm} (7)

\[\int_\Omega v \left[ -\frac{d}{dr}\left(r\frac{du}{dr}\right) + ur \right] dr = 0\]  \hspace{1cm} (8)

\[\int_\Omega r\frac{dv}{dr}\frac{du}{dr}dr - rv\frac{du}{dr}\bigg|_0^1 + \int_\Omega rvu dr = 0\]  \hspace{1cm} (9)

\[\int_\Omega r\frac{dv}{dr}\frac{du}{dr}dr + \int_\Omega rvu dr = 0\]  \hspace{1cm} (10)

Find \(u \in X^N\) s.t.

\[\int_\Omega r\frac{dv}{dr}\frac{du}{dr}dr + \int_\Omega rvu dr = 0\]  \hspace{1cm} (11)

, where \(v \in X_0^N\)

b.

For this problem an additional mass operator must be constructed to account for the second term in the equation along with a different stiffness variable:

\[nu = \text{sparse(diag(xq))}\];

\[Ab = D^\dagger*nu*B*D\];

\[Bb = Jh^\dagger*nu*B*Jh\];

with right-hand-side:

\[b = -R*(Ab+Bb)*ub\];

The computed flux is \(-0.446\) at \(r = 1\).
Here, it is clearly shown that the flux value converges. The following is the plot of the solution: