This problem set consists of several parts (two parts for the 3 credit-hour option; three parts for the 4 credit-hour option). Part I is due Wednesday, Oct. 17, 5 PM, Part II (and III, 4 cr.) due Tuesday, Oct. 23, 5 PM. Please consult with Prof. Fischer or the TA if you have questions.

In Part I, you are asked to analyze the flying performance of a taper-flat bearing under the assumption of incompressible flow. The incompressible model yields the Reynolds equation derived in class,

\[-\nabla \cdot \left( \frac{h^3}{12\mu} \right) \nabla p = -\frac{1}{2} \nabla \cdot (uh), \quad p(x, y) = 0 \quad \text{on } \partial\Omega,\]

which amounts to a 2D Poisson problem with variable coefficient determined by the bearing height, \( h(x, y) \). The domain \( \Omega \) is the subset of the \( x-y \) plane denoted by the Slider 1 pad depicted in Fig. 1. Throughout this assignment, we assume that the surface velocity is uniform in the \( x \) direction: \( u = [U, 0] \).
Figure 1: Taper-flat slider configuration: (top) plan-view for full slider pair with deep center recess, load $2F$ applied between the rails (sliders) at $x$-location $x_F$; (bottom) side-view for single slider with positive pitch, $\gamma$, load $F$, leading edge taper ($1^\circ$), disk velocity $U$, and center-of-pressure $x_p$.

Figure 1 (top) shows a plan form view of the full slider pair, which acts like a pair of skis supporting the load $F$ at $\oplus$. For our configuration (uniform $U$, no roll or yaw), it suffices to analyze a single slider of width $W$ (e.g., Slider 1). The recess region is deep enough to not generate significant bearing pressure. The total load for the pair will be $2F$ but our single-slider analysis will produce a bearing load $F$. In the rest of the discussion we consider only a single slider and will be concerned with the load point and center-of-pressure ($x_F$ and $x_p$, respectively) in the $x$ direction, only.

FEM source code for computing the pressure under a taper-flat bearing of length $L$ and width $W$, at a given flight configuration (trailing height, pitch)=$(h_2, \gamma)$, is provided. For the **Part I** of the assignment you are asked to do tasks a–d., listed below.

**a.** Verify the code against the *analytical solution* for the 1D wedge bearing problem (i.e., the “taper flat” bearing at a pitch $\gamma$, without the leading $1^\circ$ taper). Specifically, plot the maximum
Pointwise error in $p(x)$ vs. the analytical expression,

$$\tilde{p}(x) = \frac{\alpha \Lambda}{1 - \alpha^2} \left( \frac{1}{H^2} - \frac{1}{\alpha^2} \right) - \frac{\Lambda}{1 - \alpha} \left( \frac{1}{H} - \frac{1}{\alpha} \right), \quad (2)$$

where $\alpha = h_1/h_2$, $H(x) = \alpha + (1 - \alpha)(x/L)$, and $\Lambda = 6\mu UL/h_2^2$. Note that $h_1$ is the leading-edge height in the absence of the taper.

You will have to set your taper angle to zero to conduct these tests. You will also need to convert the problem to have Neumann conditions in $y$, which can be done with a single character change to the existing code.

Solution

To complete part a, change the line:

$$\text{Ta} = \pi/180; \quad \% \text{ m Taper angle (1-deg)}$$

To

$$\text{Ta} = 0; \quad \% \text{ m Taper angle (1-deg)}$$

And

$$\text{Ry} = \text{speye}(\text{Ny}+1); \text{Ry} = \text{Ry}(2:\text{end-1,:});$$

To

$$\text{Ry} = \text{speye}(\text{Ny}+1); \% \text{Ry} = \text{Ry}(2:\text{end-1,:});$$

The first change sets the taper angle to 0. The first statement of the $\text{Ry}$ assignment line sets the $Y$ restriction matrix to correspond to Neumann BC. The second statement truncates it to correspond to Dirichlet BC. Therefore, if the second statement is commented, $\text{Ry}$ will act as the Neumann restriction matrix.

The following plot is obtained for the pressure solution for the finest mesh considered:
Compared to the analytical solution, the maximum pointwise error decays to approximately $4 \times 10^{-7}$. 
b. Add a few (< 5) lines of code to compute the load, $F$ and center-of-pressure, $p$:

$$F = \int_{\Omega} p(x, y) \, dx \, dy \quad x_p = \frac{1}{F} \int_{\Omega} x \, p(x, y) \, dx \, dy.$$ 

(These integrals are easy in the FEM context.) To demonstrate that you are correctly computing $F$, verify your computed load against the analytical value $\tilde{F}$ for your wedge configuration $(L, W, h_2, \gamma)$,

$$\tilde{F} = \frac{\Lambda L W}{(1 - \alpha)^2} \left( \ln(\alpha) + 2 \frac{1 - \alpha}{1 + \alpha} \right).$$

(See http://rotorlab.tamu.edu/me626/Notes.pdf/Notes02_App_1D Bearings.pdf for a discussion of the analytical solutions.)

**Solution**

In FEM, integration of functions can be evaluated by multiplying the discretized function by a mass matrix. Recall

$$\int_{\Omega} vu \, d\Omega \approx v^T Bu$$ \hspace{1cm} (3)
Therefore,

\[ \int_\Omega p \, d\Omega \approx 1^T B p \]  

(4)

\[ \text{load} = \text{sum(sum(Bbx*p*Bby'))}; \]

and

\[ \int_\Omega x_p \, d\Omega \approx 1^T B w \]  

(5)

where \( w = x \cdot p \)

\[ \text{xbar} = \text{sum(sum(Bbx*(p.*X)*Bby'))}/\text{load}; \]

The load and \( x_p \) values obtained were: \( \text{load} = 2.366 \times 10^{-8} \) and \( x_p = 2.050 \times 10^{-3} \).

The load behavior as a function of \( N \) was compared against the analytical solution:

![Load Error vs. N](image)

and it clearly exhibits the expected \( O(N^2) \) behavior.

c. Returning to the 2D bearing problem for your given taper-flat configuration, run the code with \( N = 50 \).

Plot the pressure as a mesh plot: \texttt{mesh(X,Y,s*p);} \texttt{axis equal}. Choose a pressure scale \( s \) so that the plotted mesh profile is roughly as high as the bearing is wide. (It should look something like Fig. 2, left.)

\textbf{Answer the following:}

- What is the trailing height \( h_2 \) (m)? (This is given to you.)
• What is the pitch, $\gamma$ (radians)? (This is given to you.)
• What is the load, $F$ (N)?
• What is the center-of-pressure, $x_p$ (m)?
• What is the average pressure, $F/(LW)$, in atmospheres?

Solution

For $h_2 = 6.3 \times 10^{-7}$ and pitch = $6.3 \times 10^{-5}$, $F = 3.304 \times 10^{-2}$ N and $x_p = 2.000 \times 10^{-3}$ m. The average pressure is $\bar{p} = 1.241 \times 10^{-1}$ Pa. The following is the plot of the solution:

![Pressure Field for h2=6.3e-07, pitch=6.3e-05, and N=50 (Incompressible)](image)

To find the correct values for your particular case, use abi_load.m under the solution script.
Figure 2: Taper-flat example: (left) pressure distribution; (right) example of parameter-space search (for a compressible bearing) with target load \( F = 7 \) grams and load point \( \frac{x_p}{L} = 0.48 \).

d. Inspection of Fig. 1 indicates that the slider is subject to a net torque if \( x_F \neq x_p \). Under the pictured loading scenario, the slider will tilt forward until the center-of-pressure \( x_p \) matches the load point, \( x_F \). The slider will be in equilibrium only when \( F \) is equal to the applied load and when \( x_p \) equals the load application point, \( x_F \).

The bearing code computes \( F \) and \( x_p \) as outputs for given inputs \( h_2 \) and \( \gamma \). As an engineer, your task is to find the flying conditions \((h_2, \gamma)\) as a function of the loading conditions \((F, x_F)\). (For example, What is the flying height \( h_2 \) when the load is \( F = 0.07 \) N and is placed at \( x/L = 0.55 \)?)

There are many ways to tackle this problem. Here, you should consider taking a square search space, say, \( h_{2,i}, i = 0, \ldots, 10 \), and \( \gamma_j, j = 0, \ldots, 10 \), where the \((h_{2,i}, \gamma_j)\) values cover sufficient range to capture the target \((F, x_p)\) values in the output. An example of such a study is shown in Fig. 2 (right). With a plot like Fig. 2 (right), you can estimate the \((h_2, \gamma)\) pair corresponding to the target \((F, x_p)\) values. If the target values are not in the output range, shift the parameter range until the value of interest is inside the quadrilateral. You can then use (e.g., bilinear) interpolation to find the \((h_2, \gamma)\) values for equilibrium. Run the bearing code with those inputs to verify that your choice is close to the target (to within a few digits) of the desired load and center-of-pressure values.

**Answer the following questions for the equilibrium configuration:**

- What is the load, \( F \) (N)? (This is given to you.)
- What is the center-of-pressure, \( x_p \) (m)? (This is given to you.)
- What is the trailing height \( h_2 \) (m)?
- What is the pitch, \( \gamma \) (radians)?
- Show your parameter-space plot (like Fig. 2, right).
Solution

First find a large enough parameter ranges for $h_2$ and pitch and find what the mapping looks from $(x_p, F) \rightarrow (h_2, \gamma)$:

We can then interpolate the $h_2$ and $\gamma$ functions at the desired $x_p$ and $F$ values. Once we do, we can further narrow the parameter range around the interpolated values and repeat the process:
The following is the pressure distribution for $h_2$ and $\gamma$ that produces the desired $x_p$ and load:
To verify that this set of parameters produces the desired behavior, \texttt{abi_load.m} can be called with the interpolated $h_2$ and $\gamma$ and we get $F = 1.875000002894692 \times 10^{-2}$, and $x_{bar} = 1.306000002360156 \times 10^{-3}$. Which has accuracy that is appropriate for engineering applications.
Part II:

For the clearances and loading conditions considered in modern disk drives there are two important effects that impact the overall loading, *compressibility* and *mean-free-path effects*. In Part II, you are asked to do two things.

1. Compute $F$ and $x_p/L$ for the compressible Reynolds equation for your base configuration $(h_2, \gamma)$.

2. Compute $F$ and $x_p/L$ for the compressible Reynolds equation with the added *slip* correction that accounts for mean-free-path effects.

**Compressibility.** With peak pressures rising substantially above 1 atm, the air molecules are squeezed together, which implies that the fluid is denser between the disk and the slider than it is under ambient conditions. For the same reasons that the momentum equations are viscous dominated, the thermal transport is diffusion dominated and a reasonable assumption is that the temperature is uniform.$^1$ For air, we have the equation of state

$$p_a = \rho RT,$$

which is to say that $\rho = p_a \times constant$ for the isothermal model, where $p_a = p + p_{atm}$ is the *absolute pressure* and $p_{atm}$ is the ambient pressure (at 1 atmosphere under standard conditions). Consequently, we can modify the mass conservation equation (1) to include density, which is proportional to $p_a$.

We start with the mass conservation statement,

$$-\nabla \cdot \left( \frac{p_a h^3}{12 \mu} \right) \nabla p_a = -\frac{1}{2} \nabla \cdot \rho uh, \quad p_a(x, y) = p_{atm} \text{ on } \partial \Omega.$$  \hspace{1cm}(7)

Note the inhomogeneous Dirichlet boundary condition for the absolute pressure, which must match the ambient (atmospheric) pressure, $p_{atm}$.

Using (6) with the isothermal model, we substitute $p_a/RT$ for $\rho$ and cancel $(RT)^{-1}$ from both sides to yield the *compressible Reynolds equation* for the absolute pressure, $p_a$,

$$-\nabla \cdot \left( \frac{p_a h^3}{12 \mu} \right) \nabla p_a = -\frac{1}{2} \nabla \cdot uh p_a, \quad p_a(x, y) = p_{atm} \text{ on } \partial \Omega,$$

which is nonlinear in $p_a$. More importantly, the unknown $p_a$ also now appears on the right, which is the Couette-flow part of the mass conservation statement.

It is still convenient to split $p_a$ into the pressure variance $p$ (which we were computing in Part I) and the constant atmospheric part, $p_{atm} = 101325 N/m^2$,

$$p_a(x, y) = p(x, y) + p_{atm}.$$  \hspace{1cm}(9)

As before, we have

$$\frac{\partial p_a}{\partial x} = \frac{\partial p}{\partial x} \text{ and } \frac{\partial p_a}{\partial y} = \frac{\partial p}{\partial y},$$

$^1$The isothermal assumption does *not* hold for acoustic waves, where the state is *isentropic* rather than isothermal. There, we have $p = C \rho^\gamma$, with $\gamma$ being the specific heat ratio.
so $\nabla p = \nabla p_a$. For the Couette term on the right-hand side, however, we have

$$\frac{\partial}{\partial x} \left( \frac{Uh}{2} p_a \right) = \frac{\partial}{\partial x} \left( \frac{Uh}{2} p \right) + \frac{\partial}{\partial x} \left( \frac{Uh}{2} \right) p_{atm}.$$ \hspace{1cm}

The second term on the right leads to the same forcing term as in (1), albeit scaled by $p_{atm}$. The first term on the right is new. It is a linear (advection-like) term in the unknown $p$ that corresponds to the local density (i.e., pressure) variation being propagated by the motion of the disk underneath the slider. (For our system, $\mathbf{u} = (U, V)$, but $V = 0$, so there is no corresponding $\frac{\partial}{\partial y}$ component.)

Rearranging (8) so that unknowns are on the left and known quantities are on the right yields the compressible Reynolds equation,

$$-\nabla \cdot \left( \frac{p_a h^3}{12 \mu} \right) \nabla p + \frac{1}{2} \nabla \cdot (\mathbf{u} hp) = -\frac{p_{atm}}{2} \nabla \cdot (\mathbf{u} h),$$ \hspace{1cm}

(10)

$$p_a = p + p_{atm}$$ \hspace{1cm}

(11)

$$p = 0 \text{ on } \partial \Omega.$$ \hspace{1cm}

(12)

The weighted residual formulation for this problem is Find $p \in X^N$ such that for all $v \in X_0^N$,

$$\int_{\Omega} \nu \nabla v \cdot \nabla p \, dV - \frac{1}{2} \int_{\Omega} \nabla v \cdot (\mathbf{u} hp) \, dV = \frac{1}{2} \int_{\Omega} \nabla v \cdot (\mathbf{u} h p_{atm}) \, dV,$$ \hspace{1cm}

(13)

with

$$\nu := (p_a) h^3/(12 \mu) = (p + p_{atm}) h^3/(12 \mu).$$ \hspace{1cm}

(14)

The matrix formulation for (13) is

$$R(\bar{A} - \bar{C}) R^T p = R\bar{C}_{p_{atm}},$$ \hspace{1cm}

(15)

where $\bar{A} = \bar{A}(p_a)$ depends on $p$. Here, $\bar{p}_{atm}$ is simply $p_{atm}$ times the vector of all ones. An easy (and, in this case, convergent) strategy for treating the nonlinearity in $\bar{A}$ is to simply iterate (15), setting $p_a = p + p_{atm}$ and $\bar{A} = \bar{A}(p_a)$ after each iteration. With $p_a = p_{atm}$ as an initial guess, one needs only 5-6 iterations of (15) for the nonlinear solution to converge.

II a. (3 and 4 credit-hour) Answer the following:

- What is the trailing height $h_2$ (m)? (This is given to you.)
- What is the pitch, $\gamma$ (radians)? (This is given to you.)
- What is the load, $F$ (N)?
- What is the center-of-pressure, $x_p$ (m)?
Solution

To solve for the pressure with compressibility, we need another matrix operator that allows us to approximate the additional convection term:

\[-\frac{1}{2} \int_\Omega \nabla v \cdot (\bar{u} hp) dV = -\frac{1}{2} \int_\Omega \left( \frac{dv}{dx} U hp + \frac{dv}{dy} 0 \cdot hp \right) dV\]

\[\approx -\frac{U}{2} (D_x R^T \bar{\gamma})^T \bar{B}(JR^T \bar{p}) \]

\[= -\gamma^T \left( R \frac{U}{2} D_x^T \bar{B} JR^T \right) \bar{p} \]

\[= \gamma^T (RCR^T) \bar{p} \]

So in the code we need to define:

\[Cb = Dxt'\ast\text{sparse(diag(reshape(U*hq,nq,1)))*0.5*Bt*Jt;}\]

and set the viscosity according to the new physics:

\[\nu = \text{sparse(diag(pa.*reshape(hq.^3,nq,1)))/(12*mu);}\]

For \(h_2 = 6.3 \times 10^{-7}\) and \(\gamma = 6.3 \times 10^{-5}\), \(x_p = 1.889 \times 10^{-3}\) and \(F = 4.930 \times 10^{-2}\). The following is the plot of the pressure solution for this configuration:
II b. (4 credit-hour only): Evaluate, as you did in part (I-d), the equilibrium flying conditions for the compressible case and answer the following:

- What is the load, $F$ (N)? (This is given to you.)
- What is the center-of-pressure, $x_p$ (m)? (This is given to you.)
- What is the trailing height $h_2$ (m)?
- What is the pitch, $\gamma$ (radians)?
- Show your parameter-space plot (like Fig. 2, right).

Solution

As we did in I-d., we first choose a parameter space that contains the desired $x_p$ and load:
We again use interpolation and successively shrink the parameter space until we get an accurate solution:
Once again, we look at the output load ans $x_p$ given the interpolated pitch and $h_2$ as input: $F = 1.87499997530005 \times 10^{-2}$ and $x_p = 1.306000000722027 \times 10^{-3}$. So the interpolated values are accurate. The following is the final pressure solution:
Mean-Free-Path Effects. The mean-free-path of air molecules at standard temperature and pressure (STP) is

\[ \lambda \approx 6.8 \times 10^{-8} \text{ m}, \quad (20) \]

which is a significant fraction of the slider-bearing flying height \( h_2 \). Under such rarefied conditions, the “no-slip” boundary condition for the fluid at the disk and bearing surfaces is a less than perfect model. Researchers in tribology have developed (and validated) a simple correction to the compressible Reynolds equation that requires a simple modification to (14). Specifically, we replace (14) by

\[ \nu = \frac{1}{12\mu} \left( 6\lambda p_{atm} h^2 + (p_a)h^3 \right). \quad (21) \]

(All students.) Add the mean-free-path correction to your compressible Reynolds code and use it to answer II.c and IId.

II c. (3 and 4 credit-hour) Answer the following:

- What is the trailing height \( h_2 \) (m)? (This is given to you.)
- What is the pitch, \( \gamma \) (radians)? (This is given to you.)
- What is the load, \( F \) (N)?
- What is the center-of-pressure, \( x_p \) (m)?

Solution

For \( h_2 = 6.3 \times 10^{-7} \) and \( \gamma = 6.3 \times 10^{-5} \), \( x_p = 1.759 \times 10^{-3} \) and \( F = 3.422 \times 10^{-2} \). The following is the plot of the pressure solution for this configuration:

The following is the plot of the pressure solution:
II d. (3 and 4 credit-hour) Answer the following:

- For your given baseline configuration, how does $F$ compare for the three cases (incompressible, compressible, compressible with mean-free-path correction)?

- For your given baseline configuration, how does $x_p$ compare for the three cases.

For the three cases,

<table>
<thead>
<tr>
<th></th>
<th>Incompressible</th>
<th>Compressible</th>
<th>Mean-Free Path</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load (N)</td>
<td>$3.30 \times 10^{-2}$</td>
<td>$4.93 \times 10^{-2}$</td>
<td>$3.42 \times 10^{-2}$</td>
</tr>
<tr>
<td>$x_p$ (m)</td>
<td>$2.00 \times 10^{-3}$</td>
<td>$1.89 \times 10^{-3}$</td>
<td>$1.76 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

The compressible case had the highest load of all the cases and the incompressible case had the lowest load. The incompressible case had an error of 3.5% compared to the mean-free path case.

The incompressible case had the longest center of pressure location and the mean-free path case had the shortest center of pressure location. The incompressible case had an error of 13.6% compared to the mean-free path case.

The mathematical model for the incompressible case is the simplest of all three. The mean-free path case has the most complicated model, and one must solve for the pressure iteratively.
Although the cost of developing the code is the smallest and the time-to-solution is the shortest for the incompressible model, one must also consider the inaccuracies that arise from using a simpler model. In this particular configuration, there was a small, but significant error for both the load and the center of pressure location. If designed according to the incompressible model, the slider device may be non-functional depending on the engineering tolerance of the design.

II e. Extra No-Credit Problem; Only because it’s fun.

- For $\gamma = 0$, how does $F$ compare for the three cases (incompressible, compressible, compressible with mean-free-path correction)?

- What is going on? (Hint, plot the pressure distributions and discuss what you think is happening.)

Solution

For the three cases,

<table>
<thead>
<tr>
<th>Load (N)</th>
<th>Incompressible</th>
<th>Compressible</th>
<th>Mean-Free Path</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1.13 \times 10^{-2}$</td>
<td>$6.05 \times 10^{-2}$</td>
<td>$3.92 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

For $\gamma = 0$, the load values in decreasing order are: compressible, mean-free path, and incompressible.

One clear difference between incompressible and compressible or mean-free path solutions is that after the initial ramp up of the pressure, the pressure decays rapidly for the incompressible solution; the compressible pressure does eventually become 0 at the end, it goes under less rapid decay and then it decays rapidly at the very end of the slider. This is due to the convection term that is present in the mathematical model.

The difference between the compressible model and the mean-free path model is more subtle, but we can observe that the mean-free path solution decays to a smaller value before the boundary layer. Thus, for a comparable maximum pressure value, the integral of the pressure will be larger for the compressible case without the mean-free path correction.
Pressure Field for $h_2=6.3e-07$, pitch=0, and $N=50$ (Incompressible)
Pressure Field for $h_2 = 6.3e^{-07}$, pitch=0, and N=50 (Compressible)
Pressure Field for h2=6.3e-07, pitch=0, and N=50 (Mean-Free Path Correction)
Part III (4 credit-hour only):

One of the challenges of slider bearings is that, because of the low flying heights and tight tolerances, the surfaces must be highly polished. When they are so smooth, the surfaces will tend to stick together if loaded at a time when the disk is not moving. (Under normal conditions, the head is “parked” when power is off.) The bearing requires significant loading to get the desired stiffness, which nominally implies a large spring force $F$.

An important way to achieve stiffness without a large applied load is to have a negative-pressure slider, also known as a self-acting bearing, in which the slider pair is separated by a recessed region where subambient pressures are generated due to flow blockage near the leading edge of the slider. These self-acting bearings have many advantages: they require less external load; their flying height is less dependent on $U$ (and hence, on radial position in the disk); the recess distributions can be highly optimized to provide the desired flying height and pitch with minimal variation over a range of configurations (e.g., load, $U$, yaw, and manufacturing variations).
A simple negative-pressure configuration is the H-slider configuration of Fig. 3. The rail geometry is the same as in Part I, but there is now a pair of rails, separated by a bridge of length $H = 5W$ and width $B = W$. Behind the bridge is a recessed region of depth $h_r$, which you are to find.

Extend your compressible mean-free-path code to support this more complex geometry in the $x - y$ directions. (Please consult with Prof. Fischer or the TA if you have questions on this part.) For a load of $F=2$ grams, your base load-point $x_F$, and your originally-given $h_2$ value, answer the following questions under equilibrium conditions.

- Make a mesh plot of your H-slider pressure distribution similar to Fig. 2 (left).
- What is the load, $F$ (N)? (This is given to you.)
• What is the center-of-pressure, $x_p$? (This is given to you.)

• What is the trailing height $h_2$ (m)? (This is given to you.)

• What is the recess depth, $h_r$ (m)?

• What is the pitch, $\gamma$ (radians)?

• Describe in words the process you used to find the target configuration, verification steps you took to make certain your were on the right path, etc., as you would in a write-up to a design team at a company where people might be considering this technology as an alternative to traditional taper-flat designs. (0.5-1.5 pages should suffice, not including figures.)

Solution

For $F = 0.02N$, $x_p = 1.306 \times 10^{-3}$, and $h_2 = 6.3 \times 10^{-7}$; $\gamma = 7.11 \times 10^{-5}$ and $h_r = 3.97 \times 10^{-7}$.

To verify the code, both recess depths were set to 0, $W = \frac{0.15625}{3} \cdot L$, and $H = W$. Then the load and $x_p$ values were compared against the wedge slider code to ensure that the addition of the more complicated nodal distribution did not break the functionality of the code.

In order to find the target configuration the same methodology as before, but Newton’s Method can also be employed. Although Newton’s Method can be sensitive to initial conditions, we can start with the values that should be within 1-2 orders of magnitude of the correct solution.

The domain was discretized such that in the x-direction, $3 + N/5$ nodes were used in the taper section, $N/2$ nodes were used in the bridge section, $N$ nodes were used in the rest of the section. In the y-direction, $N/2$ nodes are used in each rail section and $N$ for the remaining section.

The following is the mesh for $N = 50$:
N = 50 was used for the initial Newton iterative solve. The initial condition was \( \gamma = 10^{-5} \) and \( h_r = 10^{-7} \) since those numbers are expected to be within an order of magnitude of the solution. The gradient of the function that needs to be minimized, \( G(\gamma, h_r) = \left( \frac{x_p - x_p^*}{F - F^*} \right) \), is approximated by a multi-variate finite difference scheme:

\[
\nabla G = \begin{bmatrix} \frac{\partial x_p}{\partial \gamma} & \frac{\partial x_p}{\partial h_r} & \frac{\partial x_p}{\partial F} \\
\frac{\partial F}{\partial \gamma} & \frac{\partial F}{\partial h_r} & \frac{\partial F}{\partial h_r} \end{bmatrix}
\]  

(22)

\begin{align*}
\frac{\partial x_p}{\partial \gamma} \bigg|_{(\gamma^k, h_r^k)} & \approx \frac{x_p(\gamma^k (1 + \epsilon), h_r^k) - x_p(\gamma^k, h_r^k)}{\epsilon} \\
\frac{\partial x_p}{\partial h_r} \bigg|_{(\gamma^k, h_r^k)} & \approx \frac{x_p(\gamma^k, h_r^k (1 + \epsilon)) - x_p(\gamma^k, h_r^k)}{\epsilon} \\
\frac{\partial F}{\partial \gamma} \bigg|_{(\gamma^k, h_r^k)} & \approx \frac{F(\gamma^k (1 + \epsilon), h_r^k) - F(\gamma^k, h_r^k)}{\epsilon} \\
\frac{\partial F}{\partial h_r} \bigg|_{(\gamma^k, h_r^k)} & \approx \frac{F(\gamma^k, h_r^k (1 + \epsilon)) - F(\gamma^k, h_r^k)}{\epsilon}
\end{align*}

(23, 24, 25, 26)
where \( \epsilon \) was chosen to be \( 10^{-8} \).

The first set of Newton iteration was terminated with 7 iterations for absolute tolerance of \( 10^{-10} \) for each component with \( x = \left( 7.089 \times 10^{-5}, 3.958 \times 10^{-7} \right) \).

The second set of Newton iterations (\( N = 100 \)) was terminated with 4 iterations for the same tolerance with \( x = \left( 7.107 \times 10^{-5}, 3.969 \times 10^{-7} \right) \).

Therefore, it is concluded that \( h_r = 3.97 \times 10^{-7} \) and \( \gamma = 7.11 \times 10^{-5} \) for the target configuration, this was verified by running the function with those values as input and the output was \( x_p = 1.306000000000018 \times 10^{-3} \) and \( F = 2.000000000000026 \times 10^{-2} \). The following is the plot of the solution:

![Pressure Field for h2=3e-07, pitch=7.1071e-05, and N=100 (H-Slider)](image_url)