\[- \nabla^2 \tilde{u} = f \quad \iff \quad \frac{\partial \tilde{u}}{\partial t} = \nabla^2 \tilde{u} + f\]

and let \( t \to \infty \).

\[ A \tilde{u} = f \]

\[ \frac{d \tilde{u}}{dt} = -A \tilde{u} + f \quad u(t=0) = \tilde{u} \]

**Idea:**
- Start with an inexpensive timesteper—Euler Forward (EF) and improve it to make convergence faster.
- Temporal accuracy is not important.

**Two (of many) ways to improve upon EF:**

(i) Multigrid

(ii) Conjugate gradients.

- Can be used together.
Want to solve $Au = f$

Say,

$$A = \frac{1}{\partial x^2} \begin{pmatrix} 2 & -1 \\ -1 & 2 \\ \vdots & \vdots \\ -1 & 2 \end{pmatrix}$$

Or

$$A = \frac{1}{w^2} \begin{pmatrix} 4 & -1 & -1 & \cdots & -1 \\ -1 & 4 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & 4 & -1 \\ -1 & -1 & \cdots & -1 & -1 \end{pmatrix}$$
\[
\frac{u_k - u_{k-1}}{\Delta t} = -A u_{k-1} + f \quad u^0 = 0
\]

\[
u_k = u_{k-1} + \Delta t \left( f - A u_k \right)
\]

- At steady state, \( k \to \infty \) \( f = Au_k \)

- Max \( \Delta t \) from stability

\[
\begin{align*}
1D: \quad \Delta t & \leq \frac{h^2}{2V} \quad (\nu = 1) \\
2D: \quad \Delta t & \leq \frac{1}{2} \frac{h^2}{2D} = \frac{h^2}{2V}
\end{align*}
\]

Ding (A); \( D = \frac{2}{h^2} I \) \( 1D \)

\( D = \frac{u}{h^2} I \) \( 2D \)
Let's talk about the "smoother" first.

\[ u_k = u_{k-1} + \sigma N^T \hat{r}_{k-1} \quad \text{ Computable!} \]

Recall, the residual satisfies
\[ r_{k-1} := f - Au_{k-1} \]
\[ = Au - A u_{k-1} \]
\[ = A(u - u_{k-1}) = A e_{k-1} \]

where
\[ e_{k-1} := u - u_{k-1} \] is the error vector,
The error equation:

Analysis (what is happening to the error, not what we can compute, in general):

- $u_k$: iterate - exact (unknown) solution.

$$
\begin{bmatrix}
  u_k \\
  u
\end{bmatrix} =
\begin{bmatrix}
  u_{k-1} + \delta m^{-1}(f - A u_{k-1}) \\
  u
\end{bmatrix}
$$

$$
\begin{align*}
  e_k &= e_{k-1} - \delta m^{-1} A e_{k-1} \\
  &= (I - \delta m^{-1} A) e_{k-1} \\
  &= G e_{k-1}
\end{align*}
$$

* Error will reduce if spectral radius, $\rho(G) < 1$. 
In case of EF for Poison, error satisfies

\[ e_k = e_{k-1} \cdot e_l \cdot e_{k-1} \quad \text{with} \quad L = -A \]

\[ = (I + \Delta t L) \cdot e_{k-1} \]

\[ \Delta t = \sigma \frac{\Delta x^2}{2\nu} \quad (\nu=1) \]
Notice if we choose \( \Delta t = \Delta t_{\text{max}} = \frac{A_x^2}{2} \) (i.e., \( \sigma = 1 \)) then highest wave numbers in the error are mildly damped by EF.

If you want these modes to be heavily damped, use \( \sigma = \frac{1}{2} \Rightarrow \Delta t = \frac{\Delta t_{\text{max}}}{2} \).
Summary

- Smoothing

- Damped Jacobi \( (\sigma = \frac{1}{2} - \frac{2}{3}) \)
  leads to \( \varepsilon_k \) being very smooth
  after just a few iterations. (!)

Next Step - Coarse Grid Correction

- Recall: \( A \varepsilon_k = \delta_k = f - A u_k \)
  \( \delta_k = \)

- If we find \( \varepsilon_k \), set
  \( u_k = u_k + \varepsilon_k \) and return. DAVE
12. Coarse Grid Correction

- On the surface

\[ A e_k = f_k \]

- Seems as hard as \( Au = f \).

- However, we know an important property: \( e \) is smooth - without knowing \( e \).

- Therefore, can approximate \( e(x) \) by a low-dimensional interpolant.
Let \( J \) be interpolation matrix - e.g.,

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots \\
\vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
\end{bmatrix}
\]

last - interp. add, m,

minus first & last columns & rows.
In same spirit as Galerkin method, approximate $e$ by $\tilde{e}_f := J e_c$

$$A J e_c = \tilde{e}_f \leftrightarrow N \text{-} 1 \text{ equations}$$

$$= \frac{N - 1}{2} \text{ unknowns} \quad (N \text{ even})$$

Reduce # of equations by solving:

(i) $$(J^T A J) e_c = J^T r_k$$

(ii) $e_f = J e_c$

Easy to show that:

$$\| e_f - e \|_A \leq \| w - e \|_A \quad A w \in \mathcal{R}(J)$$

(A - spd)
Overall scheme:

\[ u = 0 \quad r = f \]

for \( k = 1, \ldots \)

\[ s = \sigma m^{-1} r \]

\[ u = u + s \]

\[ r = f - Au = r - As \]

ends

\[ r_c = J^{-1} r \]

\[ e_c = A_c^{-1} r_c \]

\[ e_f = J e_c \]

\[ u = u + e_f \]

ends

If \( \| r \| < tol \), break.