Problem 1

A two-point Gauss-Legendre quadrature rule, $I_Q(f)$ is used to approximate

$$ I := \int_0^1 f(x) \, dx. $$

What is the error, $|I - I_Q|$, for each of the following functions?

- a. $f = x^0$
- b. $f = x^1$
- c. $f = x^2$
- d. $f = x^3$
- e. $f = x^4$

Solution

For GL nodes, the integration is exact for integrand in $P_{2n+1}$. Therefore, the error for (a), (b), (c), and (d) are 0 since $n = 1$:

$$ |I(1) - I_Q(1)| = 0 $$
$$ |I(x) - I_Q(x)| = 0 $$
$$ |I(x^2) - I_Q(x^2)| = 0 $$
$$ |I(x^3) - I_Q(x^3)| = 0 $$
For \( f = x^4 \), the integration is **not exact** as \( x^4 \) is not in \( \mathbb{P}_{2n+1} \). The 2-Point GL integration has evaluation points \( \hat{x} = -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \) and the weights \( \hat{w} = 1, 1 \) in \( \hat{\Omega} = [-1, 1] \). So for this problem in interval \([0, 1]\), the evaluation points are \( x = \frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{2} + \frac{1}{2\sqrt{3}} \) and weights are \( w = 1/2, 1/2 \). For (e),

\[
\begin{align*}
\hat{f} \left( \frac{1}{2} - \frac{1}{2\sqrt{3}} \right) = \frac{7-4\sqrt{3}}{36} \quad \text{and} \quad \hat{f} \left( \frac{1}{2} + \frac{1}{2\sqrt{3}} \right) = \frac{7+4\sqrt{3}}{36}.
\end{align*}
\]

Therefore, \( I_Q = \frac{7}{36} \) for this monomial, and

\[
|I(x^4) - I_Q(x^4)| = |1/5 - 7/36| = 1/180 \approx 0.00556.
\]

**Problem 2**

The two-point boundary value problem

\[
-\nu \frac{d^2u}{dx^2} + \beta u = f, \quad u(0) = u(1) = 1
\]

is discretized with linear finite elements on \( \Omega = [0, 1] \), leading to a system for the unknowns of the form

\[
\nu A \mathbf{u} + \beta B \mathbf{u} = \mathbf{b}.
\]

For the case of two elements with nodal points \( x_0 = 0, x_1 = 1/2, \) and \( x_2 = 1 \), the extended stiffness and mass matrices, \( \bar{A} \) and \( \bar{B} \), respectively, are each \( 3 \times 3 \).

**a.** What is the entry in the \((3, 3)\) position for \( \bar{A} \)?

**b.** What is the entry in the \((3, 3)\) position for \( \bar{B} \)?

**Solution**

To find the entries in \( \bar{A} \) and \( \bar{B} \), it helps to understand what the basis functions look like:
a.
To find $\bar{A}_{3,3}$, we must integrate the square of derivative of $\phi_2$:

$\bar{A}_{3,3}$ is the area under the $\frac{d\phi_2}{dx}^2$ curve:
\[ \bar{A}_{3,3} = \int_0^1 \frac{d\phi_2^2}{dx} \, dx \]
\[ = \int_0^{1/2} 0^2 \, dx + \int_{1/2}^1 2^2 \, dx \]
\[ = 2 \]

b.

To find \( \bar{B}_{3,3} \), we must integrate the square of \( \phi_2 \):

\( \bar{B}_{3,3} \) is the area under the \( \phi_2^2 \) curve.
\[B_{3,3} = \int_0^1 \phi_2^2 \, dx\]
\[= \int_0^{1/2} 0^2 \, dx + \int_{1/2}^1 (2(x - 1/2))^2 \, dx\]
\[= 4 \int_0^{1/2} y^2 \, dy\]
\[= 4 \cdot \frac{1}{24}\]
\[= \frac{1}{6}\]