1. A 2nd-order centered difference approximation is used to discretize
\[-\frac{d^2\bar{u}}{dx^2} = f, \quad \bar{u}(0) = \bar{u}(2) = 0,
\]
resulting in the system
\[Au = f.\]

(a) Assuming uniform gridspacing \(\Delta x = L/N\) with \(N = 4\), what is the smallest eigenvalue of \(A\) (to 5 digits)?

(b) What is the largest eigenvalue of \(A\) (to 5 digits)?

(c) To 5 digits, what is asymptotic value of the smallest eigenvalue of \(A\) as \(N\) is increased?

(d) The largest eigenvalue of \(A\) scales like \(O(N^q)\). What is \(q\)?

Solution

The analytical form of the discrete eigenvalues is:
\[\lambda_k = \frac{2}{\Delta x^2} \left(1 - \cos\left(\frac{k\pi\Delta x}{L}\right)\right)\]

(a) & (b)

For \(N = 4\), \(k_{\text{min}} = 1\) and \(k_{\text{max}} = 3\) correspond to the smallest and largest eigenvalue; therefore,

\[\lambda_1 = 8(1 - \cos(\pi/4)) \approx 2.3432, \quad \text{and}\]
\[\lambda_3 = 8(1 - \cos(3\pi/4)) \approx 13.657.\]

(c)

Substitute the Maclaurin series of cosine into the eigenvalue expression:
\[\lambda_k = \frac{2}{\Delta x^2} \left(1 - (1 - (k\pi\Delta x/L)^2/2) + O(\Delta x^4))\right)\]
\[= k^2\pi^2/L^2 + O(\Delta x^2)\]
\[\Rightarrow \lim_{\Delta x \to 0} \lambda_1 = \frac{\pi^2}{4} \approx 2.4674\]
(d)
Use the angle difference identity for the cosine term of the largest eigenvalue:

\[
\cos \left( \frac{N-1}{L} \frac{\pi}{N} \right) = \cos \left( \frac{N-1}{N} \pi \right) \\
= \cos(\pi - \frac{\pi}{N}) \\
= \cos(\pi) \cos \left( \frac{\pi}{N} \right) + \sin(\pi) \sin \left( \frac{\pi}{N} \right) \\
= -\cos \left( \frac{\pi}{N} \right)
\]

Now substitute this expression into the eigenvalue expression and expand cosine:

\[
\lambda_{N-1} = \frac{2}{\Delta x^2} \left( 1 - (-1 - O(N^{-2})) \right) \\
= \frac{2N^2}{L^2} (2 - O(N^{-2})) \\
= \frac{4}{L^2} N^2 - O(1) \\
\sim N^2
\]

Therefore, \( q \) is 2.

2. Consider the following two-point BVP

\[-\frac{d}{dx} \nu \frac{dp}{dx} = -\frac{dh}{dx}, \quad p(-1) = p(1) = 0,\]

with \( \nu = 0.1h^3(x) \), \( h(x) = .02 \) for \( x < 0 \), and \( h(x) = .01 \) for \( x > 0 \).

(a) Solve this problem for \( p(0) \) using linear finite elements with two elements of size \( \Delta x = 1 \).

(b) What is the lift in this case?

**Solution**

(a)
First, substitute the right-hand-side contribution to the residual by integration by parts:

\[
- \int_{\Omega} \nu \frac{dh}{dx} dx = \int_{\Omega} \frac{dv}{dx} h dx - v h \bigg|_{-1}^{1} \\
= \int_{\Omega} \frac{dv}{dx} h dx \\
= (DRv)^T \tilde{B}h \\
= v^T RD^T \tilde{B}h
\]
Modify the code from homework 2 by defining the right-hand-side vector \( \underline{f} \) to:

\[
\underline{f} = RD^T \tilde{Bh}
\]

and assigning an appropriate expression for \( \tilde{\nu} \).

Then one can obtain the value at \( x = 0 \), \( p(0) \approx 11111 \).

The following is the plot of the FEM solution:

(b) To find the lift, find the area under the \( p(x) \) curve. Since it is a triangle, the area is \( f_{\text{lift}} = \left( \frac{1}{2} \right) (2)p(0) \approx 11111 \).