TAM 570 / CSE 556: Spectral Methods for CFD

Course Overview

The objective of this course is to have students develop their own spectral-based Navier-Stokes solvers and to use these to solve interesting turbulent thermal-fluids problems in 3D. We will also cover the basic theory of spectral methods and identify why the are efficient, why they work, when they are rapidly convergent, and when they are not. Exercises are devised to allow the student to develop some fundamental coding tools, to explore the properties of spectral methods and to explore the properties of incompressible flow simulations in general.

All exercises and projects will be written in matlab. Students should bring their laptops to class as we will typically use some of the lecture period to do some coding in which the class will break up into small teams to tackle exercises for the topic of that particular lecture. Students should be proficient with a text editor (emacs/VI/etc.) so that they can develop short matlab scripts during class with minimal effort.

Course Outline

1. Introduction
   - Scales of Fluid Motion
     - Turbulence — range of scales: $Re^{\frac{9}{4}}$ scaling.
     - 3D Fourier spectral code examples (Orszag '80; PK Yeung '19).
   - Spectral methods: Objectives / Examples
     - Rapid convergence
     - Fast operator evaluation
     - Fast operator inversion
   - Equations: Navier-Stokes, Energy Transport, Model Problems
     - Navier-Stokes
       * vector notation
       * indicial notation
     - Model problems:
       * Oseen
       * Leray-regularized NS
       * Unsteady / Steady Stokes
       * Advection-diffusion
       * Poisson
       * Advection-dominated flows and the Reynolds/Peclet numbers
         · analytical solution to advection-diffusion
         · exact solution to 2nd-order finite difference problem
         · singular perturbation example
         · Reynolds/Peclet exercises.
     - Complexity due to nonlinearity and geometry
       * Nonlinearity example: Frisch
         · impact of round-off, finite precision arithmetic, conditioning
         · cases where round-off effects matter
         · cases where round-off is less important
       * 3D geometry examples:
         · Complex example.
         · Simple example–conduction in a cylinder:
           Closed-form solution & Gibbs phenomenon.
         · Cylinder exercise.
   - 3D Interpolation / differentiation complexity
     - Per-point interpolation costs in 1D/2D/3D/
     - Tensor-interpolation costs in 1D/2D/3D/
     - Interpolation exercises.

2. Numerical Basics – Spectral solutions of model problems.

- Simulation illustration:

  1D periodic wave equation with FD, Fourier, Legendre.

- 1D approximation via interpolation/projection
  - Polynomial and Fourier: FFT (exercise)
  - Fourier projection/interpolation/aliasing
  - Exponential convergence
  - Not Exponential convergence
  - Polynomial approximation
  - Orthogonal polynomials
  - Gauss quadrature
  - Exercises.
- Numerical differentiation
  - Polynomial differentation–differentiation matrix.
  - Fourier differentiation

    .Compare cost of FFT differentiation vs GLL differentiation in 1D as a function of N

    .Compare cost of FFT differentiation vs GLL differentiation in 3D as a function of N
  - Differentiation in deformed geometries – chain rule
  - Gordon-Hall mappings.
  - Exercise: Compute ∇u in a deformed domain.
  - Exercise: Find max $\|\nabla u\|$ in a deformed domain.
- Numerical integration
  - GL / GLL quadrature
  - Trapezoidal rule and Fourier methods
  - Integration in deformed geometries – chain rule
  - Exercises
    - Use Gordon-Hall to map $\hat{\Omega}$ to circular-wedge domain.
    - What is the area for the circular-wedge domain?
    - What is the rate of convergence when GLL quadrature is used to compute this area?

3. Galerkin projection for heat equation
   - 1D: Dirichlet
     - formulation / restriction matrices
     - Neumann
     - Robin
   - 2D: Dirichlet
     - Tensor- (Kronecker-)product forms
     - Product rules for multiplication of Kronecker-product matrices
     - Fast tensor-product operator evaluation
     - Fast solvers
     - Neumann
     - Robin
   - 3D: Dirichlet
     - Tensor- (Kronecker-)product forms
     - Product rules for multiplication of Kronecker-product matrices
     - Fast tensor-product operator evaluation
     - Fast solvers
   - 2D and 3D deformed geometry: iterative solvers
     - Conjugate gradient iteration
     - Preconditioning
     - Tensor-product preconditioning
     - FEM preconditioning (Orszag 80, Canuto 2011, Bello-Maldonado 2018)

4. Unsteady advection/diffusion in 1D
   - Advection
     - Finite difference
     - Fourier
     - Spectral element
     - Timestepper choices:
       * AB3
       * EF (aka AB1)
       * RK4
       * BDF3/EXT3
     - Dispersion behavior
     - CFL stability criterion.
       * skew symmetric forms
       * upwinding = skew symmetry + diffusion
   - Advection-Diffusion
     - Timestepper choices:
     - AB3/Crank-Nicolson
     - BDF3/EXT3
     - RK4
   - HW: Solve Burgers equation with Fourier, Legendre spectral, and finite differences.

     Compare with published results.

5. Unsteady advection/diffusion in 2D
   - Heat equation: Fourier $\times$ Legendre
   - Advection-diffusion: Fourier $\times$ Legendre; Fourier $\times$ Legendre + domain deformation
   - Advection-diffusion: Legendre $\times$ Legendre; Legendre $\times$ Legendre + domain deformation
   - Importance of dealiasing
   - Importance of proper outflow treatment–Dirichlet vs. Neumann.
   - CFL stability consideration.
   - HW:
     - Solve advection-diffusion in 2D channel flow Fourier $\times$ Legendre
       Check importance of dealiasing.
     - Solve advection-diffusion in 2D Taylor flow Fourier $\times$ Legendre
       Check importance of dealiasing.

6. Spectral methods for unsteady Navier-Stokes
   - Fourier $\times$ Legendre
   - Legendre $\times$ Legendre
   - Fourier $\times$ Legendre + domain deformation
   - Legendre $\times$ Legendre + domain deformation
   - HW:
     - Solve NS for Walsh eddy problem Legendre $\times$ Legendre -check importance of dealiasing
     - Solve NS for Walsh eddy problem Fourier $\times$ Fourier -check importance of dealiasing
     - Solve NS for channel flow, Fourier $\times$ Legendre
     - Anysis of the Orr-Sommerfield stability problem

7. Multi-domain spectral methods (spectral elements)
   - Heat equation: assembly operations to enforce continuity
   - Iterative solvers:
     - conjugate gradients
     - preconditioning strategies: 2D and 3D
     - projection in time
   - Unsteady advection-diffusion equation
   - Unsteady Navier-Stokes
     - $\mathbb{P}_N - \mathbb{P}_{N-2}$ formulation
     - $\mathbb{P}_N - \mathbb{P}_N$ formulation

8. SEM for 3D Navier-Stokes
   - Stabilization
     - filtering
     - artificial viscosity

9. Other topics
   - Characteristics methods
   - Turbulent outflow conditions
   - Potential flow
   - etc.