TAM 570 / CSE 556: Spectral Methods for CFD

Course Notes

There are online notes under *Notes, Papers, Books* that will be updated throughout the term. Please check these throughout the semester as most of the lecture material will be written there.

The lectures will be live and we will work in class on the quiz/homework material at the end of each lecture.

Course Overview

The objective of this course is two-fold. First, students will learn fundamental principles of spectral and spectral element methods and will use these to develop their own codes capable of solving interesting flow and heat-transfer problems. We will cover the basic theory of spectral methods and identify why the are efficient, why they work, when they are rapidly convergent, and when they are not. Second, students will learn how to apply high-order methods to thermal-fluids applications in complex domains by using state-of-the-art production-level codes. Meshing, boundary conditions, solution verification, and performance will be major components of this aspect.

Exercises are designed to allow the student to develop fundamental coding tools, to explore the properties of spectral methods, and to explore the properties of incompressible flow simulations in general. **Please bring laptops to class** as we will typically use some of the lecture period to do some coding in which the class will break up into small teams to tackle exercises for the topic of that particular lecture.

On the applied side, we will use the open source code Nek5000/RS for some of the more challenging fluid/thermal applications. The idea is to illustrate the potential of these methods when implemented with high performance as a principal objective and to allow students to address interesting flow problems.

Much of the programming and quiz work will be carried out in class, usually during the last 20–30 minutes of class time.

Course Outline

1. Introduction
   - Scales of Fluid Motion
     - Turbulence — range of scales: $Re^{\frac{9}{4}}$ scaling.
     - 3D Fourier spectral code examples (Orszag '80; PK Yeung '19).
   - Spectral methods: Objectives / Examples
     - Rapid convergence
     - Fast operator evaluation
     - Fast operator inversion
   - Equations: Navier-Stokes, Energy Transport, Model Problems
     - Navier-Stokes
       - vector notation
       - indicial notation
     - Model problems:
       - Oseen
       - Leray-regularized NS
       - Unsteady / Steady Stokes
       - Advection-diffusion
       - Poisson
       - Advection-dominated flows and the Reynolds/Peclet numbers
         - analytical solution to advection-diffusion
         - exact solution to 2nd-order finite difference problem
         - singular perturbation example
         - Reynolds/Peclet exercises.
     - Complexity due to geometry and nonlinearity
       - 3D geometry illustrations
       - Nonlinearity example: Frisch
         - impact of round-off, finite precision arithmetic, conditioning
         - cases where round-off effects matter
         - cases where round-off is less important

2. Numerical Basics – Spectral solutions of model problems.
   - Simulation illustration:
     1D periodic wave equation with finite-difference, Fourier, Legendre.
   - 1D approximation via interpolation/projection
     - Polynomial and Fourier: FFT (exercise)
     - Fourier projection/interpolation/aliasing
     - Exponential convergence
     - Not Exponential convergence
     - Polynomial approximation
     - Orthogonal polynomials
     - Gauss quadrature
     - Exercises.
   - Numerical differentiation
     - Polynomial differentation–differentiation matrix.
     - Fourier differentiation
       .Compare cost of FFT differentiation vs GLL differentiation in 1D as a function of N
       .Compare cost of FFT differentiation vs GLL differentiation in 3D as a function of N
     - Differentiation in deformed geometries – chain rule
     - Gordon-Hall mappings. **(later)**
     - Exercise: Compute ∇u in a deformed domain.
   - Numerical integration - I.
     - GL / GLL quadrature
     - Trapezoidal rule and Fourier methods
     - Exercises:
       * What is rate of convergence for trapezoidal rule when f(x)=Gaussian on [-10:10]?
       * What is rate of convergence for trapezoidal rule when f(x)=Gaussian on [0:10]?
       * What is rate of convergence for GLL when f(x)=Gaussian on [-10:10]?
       * PLOT error vs. N for each of these cases.

3. Galerkin projection for heat equation
   - 1D: Dirichlet
     - formulation / restriction matrices
     - Neumann
     - Robin
   - 2D: Dirichlet
     - Tensor- (Kronecker-)product forms
     - Product rules for multiplication of Kronecker-product matrices
     - Fast tensor-product operator evaluation
     - Fast solvers
     - Neumann
     - Robin
   - 3D: Dirichlet
     - Tensor- (Kronecker-)product forms
     - Product rules for multiplication of Kronecker-product matrices
     - Fast tensor-product operator evaluation
     - Fast solvers
   - 2D and 3D deformed geometry: iterative solvers **(later)**
     - Conjugate gradient iteration
     - Preconditioning
     - Tensor-product preconditioning
     - FEM preconditioning (Orszag 80, Canuto 2011, Bello-Maldonado 2018)

4. Unsteady advection/diffusion in 1D
   - Advection
     - Finite difference
     - Fourier
     - Spectral element
     - Timestepper choices: EB-EF, CN-AB3, RK4, BDF3/EXT3
     - Dispersion behavior
     - CFL stability criterion.
       * skew symmetric forms
       * upwinding = skew symmetry + diffusion
   - Advection-Diffusion
     - Timestepper choices:
     - AB3/Crank-Nicolson
     - BDF3/EXT3
     - RK4
   - HW: Solve Burgers equation with Fourier, Legendre spectral, and finite differences. Compare with published results. **(later)**

5. Unsteady advection/diffusion in 2D
   - Heat equation: SEM
     - assembly operations to enforce continuity
     - conjugate gradients
     - preconditioning strategies: 2D and 3D
   - Unsteady advection-diffusion equation
     - Projection in time
     - Potential pitfalls
       * Importance of dealiasing
       * Importance of proper outflow treatment–Dirichlet vs. Neumann.
       * CFL stability consideration.
   - HW:
     - Solve advection-diffusion in 2D channel flow. Check importance of dealiasing.
     - Solve advection-diffusion in 2D Taylor flow. Check importance of dealiasing.

6. Spectral methods for unsteady Navier-Stokes I
    - $\mathbb{P}_N - \mathbb{P}_{N-2}$ formulation (1 lecture)
    - $\mathbb{P}_N - \mathbb{P}_N$ formulation (2 lectures)
    - HW:
        - Solve NS for Walsh eddy problem Legendre $\times$ Legendre -check importance of dealiasing
        - Solve NS for Walsh eddy problem Fourier $\times$ Fourier -check importance of dealiasing
        - Solve NS for channel flow, Fourier $\times$ Legendre
        - Analysis of the Orr-Sommerfeld stability problem
    - Stabilization
        - filtering
        - artificial viscosity

7. Spectral methods for unsteady Navier-Stokes II
    - Numerical integration in complex domains
        - Integration in deformed geometries – chain rule
        - Exercises
            * Use Gordon-Hall to map $\hat{\Omega}$ to circular-wedge domain.
            * What is the area for the circular-wedge domain?
            * What is the rate of convergence when GLL quadrature is used to compute this area?

8. Spectral methods for unsteady Navier-Stokes III
    - Characteristics methods
    - Turbulent outflow conditions
    - Potential flow
    - etc.

9. Convergence theory for spectral methods - Fourier and Polynomial